LINEAR CORRELATIONS OF MULTIPLICATIVE FUNCTIONS

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ABSTRACT. We prove a Green–Tao theorem for multiplicative functions.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to establish an asymptotic result for correlations of real-valued multiplicative functions. Throughout this paper we write

$$S_h(x) = \frac{1}{x} \sum_{1 \leqslant n \leqslant x} h(n) \qquad \text{and} \qquad S_h(x; q, a) = \frac{q}{x} \sum_{\substack{1 \leqslant n \leqslant x \\ n \equiv a \ (\mathrm{mod}\ q)}} h(n)$$

for any $q, a \in \mathbb{Z}$, $q \neq 0$ and $x \geqslant 1$. We begin by describing the class $\mathscr{M}$ of multiplicative functions that our result applies to.

Definition 1.1. Let $\mathscr{M}$ be the class of multiplicative functions $h : \mathbb{N} \to \mathbb{R}$ such that:

(i) There is a constant $H \geqslant 1$, depending on h, such that $|h(p^k)| \leqslant H^k$ for all primes p and all integers $k \geqslant 1$.

(ii) $|h(n)| \ll_\varepsilon n^\varepsilon$ for all $n \in \mathbb{N}$ and $\varepsilon > 0$,

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(iii) There is a positive constant α_h such that

$$\frac{1}{x}\sum_{p\leqslant x}|h(p)|\log p \geqslant \alpha_h$$

for all sufficiently large x.

(iv) *Stable mean value in arithmetic progressions:*
Let $C>0$ be any constant, suppose that $x' \in (x(\log x)^{-C}, x)$ and that x is sufficiently large, and let $A \pmod q$ be any progression with $1 \leqslant q < (\log x)^C$ and $\gcd(q,A)=1$. Then

$$S_f(x';q,A) = S_f(x;q,A) + O\Bigg(\varphi(x)\frac{q}{\phi(q)}\frac{1}{\log x}\prod_{\substack{p\leqslant x\\ p\nmid q}}\left(1+\frac{|f(p)|}{p}\right)\Bigg)$$

for some function φ with $\varphi(x)\to 0$ as $x\to\infty$.

Remark. Out of the four conditions above, the last one is certainly the most difficult to check in any application. If, however $\alpha_h > 2H/\pi$ in (iii), or if f is non-negative and $\alpha_h > H/\pi$, then condition (iv) follows from the other three conditions. The way to prove this is as follows. Let g be the multiplicative functions defined by

$$g(p^k) = \begin{cases} h(p)/H & \text{if } k=1\\ 0 & \text{if } k>1\end{cases}. \tag{1.1}$$

Then $h = g^{*H} * g'$ for some multiplicative function g' that satisfies $g'(p)=0$ at all primes, and for which bounds as in Definition 1.1 (i) hold. Since g is bounded, we may employ the Lipschitz bounds [8] of Granville and Soundararajan, which will come with an acceptable error term due to the lower bound on α_h/H. Condition (iv) then follows by combining these Lipschitz bounds with the 'pretentious large sieve' (see Proposition 8.4). We will carry out this approach in detail in Section 11 to show that $\mathscr{M}$ contains the functions $n\mapsto|\lambda_f(n)|$, where $\lambda_f(n)$ is the normalised Fourier coefficient at n of a primitive cusp form f. Another interesting example of a function for which α_h is sufficiently large is the characteristic function of sums of two squares for which (iii) clearly holds with $\alpha_h=\frac{1}{2}$. Alternatively, the Dirichlet series attached to h and its twists by characters can be considered.

The quantities $S_h(x;q,a)$ will play an important role in the statement of our main result. Since the mean value $S_h(x;q,a)$ is likely to show a very irregular behaviour for small values of q, we work with a W-trick. To set this up, let $w:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ be any function such that

$$\frac{\log\log x}{\log\log\log x} < w(x) \leqslant \log\log x$$

for all sufficiently large x, and define

$$W(x) = \prod_{p\leqslant w(x)} p.$$

To avoid the contributions from small prime factors, we will largely work in the progressions $A \pmod{W(x)}$ with $\gcd(A,W)=1$, and in suitable subprogressions thereof.

Our main result is the following.

Theorem 1.2. *Let $h_1,\dots,h_r \in \mathscr{M}$ be multiplicative functions. Let $T>1$ be an integer parameter and let $\varphi_1,\dots,\varphi_r \in \mathbb{Z}[X_1,\dots X_s]$ be linear polynomials whose non-constant parts are pairwise independent over $\mathbb{Q}$ and have coefficients that are independent of T, while the constant coefficients $h_i(0)$ may depend on T with the constraint that $h_i(0)=O(T)$. Suppose that $\mathfrak{K}\subset[-1,1]^s$ is a convex body with $\operatorname{vol}(\mathfrak{K})\gg 1$ such that $\varphi_i(T\mathfrak{K})\subset[1,T]$ for each $1\leqslant i\leqslant r$.*

Then there are positive constants B_1 and B_2 and a function $\widetilde{W}:\mathbb{R}_{>0}\to\mathbb{N}$ that takes values of the form $\widetilde{W}(x)=\prod_{p<w(x)}p^{\alpha(p,x)}$ with $\alpha(p,x)\in\mathbb{N}$ and satisfies the bound $\widetilde{W}(x)\leqslant(\log x)^{B_1}$ for all x, and that is such that the following asymptotic holds as $T\to\infty$:

$$\frac{1}{\operatorname{vol} T\mathfrak{K}}\sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}}\prod_{i=1}^{r}h_i(\varphi_i(\mathbf{n})) \;=$$

$$\sum_{\substack{w_1,\dots,w_r\\ p\mid w_i\Rightarrow p<w(T)\\ w_i\leqslant(\log T)^{B_2}}}\;\sum_{\substack{A_1,\dots,A_r\\ \in(\mathbb{Z}/\widetilde{W}\mathbb{Z})^*}}\left(\prod_{i=1}^{r}h_i(w_i)S_{h_i}\Big(T;\widetilde{W},A_i\Big)\right)\frac{1}{(w\widetilde{W})^s}\sum_{\substack{\mathbf{v}\in\\(\mathbb{Z}/w\widetilde{W}\mathbb{Z})^s}}\prod_{j=1}^{r}\mathbf{1}_{\varphi_j(\mathbf{v})\equiv w_jA_j \pmod{w_j\widetilde{W}}}$$

$$+\,o\left(\frac{1}{(\log T)^r}\prod_{j=1}^{r}\prod_{p\leqslant T}\left(1+\frac{|h_i(p)|}{p}\right)\right)\;, \tag{1.2}$$

where $w=\operatorname{lcm}(w_1,\dots,w_r)$ and $\widetilde{W}=\widetilde{W}(T)$. The error term dominates as soon as one of the functions h_i satisfies $|S_{h_i}(x)|=o(S_{|h_i|}(x))$.

Unsurprisingly, the above result can be reformulated in terms of character sums. By means of the pretentious large sieve (see Proposition 8.4), it is moreover possible to restrict these character sums to a small number of characters that have a large correlation with the h_i. A result of Elliott (Lemma 1.5 below) which will be discussed in the sequel allows one to even further restrict these character sums. If $h_i(n)=\mathbf{1}*\chi_4(n)=\frac{1}{4}r(n)$, the relevant characters would, for instance, be those induced by $\mathbf{1}$ and χ_4. The following is such a reformulation of Theorem 1.2 in terms of finite character sums:

Theorem 1.3. *With the assumptions of Theorem 1.2 in place, define for each $i\in\{1,\dots,r\}$ the integer $k_i=1+\lceil\alpha_{h_i}^{-2}\rceil$, where α_{h_i} is as in Definition 1.1. Consider for each i the set of primitive characters of conductor at most $(\log T)^{B_1}$ and enumerate them as $\chi_1^{(i)},\chi_2^{(i)},\dots$ in such way that the averages $|\sum_{n\leqslant T}\chi_j^{(i)}(n)h_i(n)|$ are in non-increasing order as j increases. Let $\mathscr{E}_i$ be the subset of characters χ modulo $\widetilde{W}(T)$ that are induced from $\chi_1^{(i)},\chi_2^{(i)},\dots,\chi_{k_i}^{(i)}$ and that have the property that*

$$\kappa(T)\sum_{n\leqslant T}|h_i(n)|\ll\Big|\sum_{n\leqslant T}\chi(n)h_i(n)\Big|,$$

for some fixed function $\kappa : \mathbb{N} \to \mathbb{R}_{>0}$ that we are free to choose and which we assume to satisfy $\kappa(x) = o(1)$. Then

$$\frac{1}{\operatorname{vol} T\mathfrak{K}} \sum_{\mathbf{n} \in \mathbb{Z}^s \cap T\mathfrak{K}} \prod_{i=1}^{r} h_i(\varphi_i(\mathbf{n})) = \beta_\infty \sum_{\substack{\chi_1, \dots, \chi_r \\ \chi_j \in \mathscr{E}_j}} \prod_{p \leqslant T} \beta_p(\chi_1, \dots, \chi_r) + o\left(\frac{1}{(\log T)^r} \prod_{j=1}^{r} \prod_{p \leqslant T} \left(1 + \frac{|h_i(p)|}{p}\right)\right),$$

where $\beta_\infty = \prod_{i=1}^r S_{|h_i|}(T)$, while the local factor at p takes the form

$$\beta_p(\chi_1, \dots, \chi_r) = \lim_{m \to \infty} \frac{1}{p^{ms}} \sum_{\mathbf{a} \in \mathbb{N}_0} \sum_{\substack{\mathbf{v} \in (\mathbb{Z}/p^m\mathbb{Z})^s \\ v_p(\varphi_i(\mathbf{v})) = a_i}} \prod_{j=1}^{r} h_j(p^{a_j}) \chi_j^*(\varphi_j(v), p^{a_i}) \left(1 - \frac{1}{p}\right)^{-1} \left(1 + \frac{|h_i(p)|}{p} + \dots\right)^{-1}$$

with

$$\chi_j^*(\varphi_j(v), p^{a_i}) = \begin{cases} \overline{\chi_{j,p}} \left(\frac{\varphi_j(v)}{p^{a_j}}\right) \left(\prod_{q \neq p,\ q \text{ prime}} \chi_{j,q}(p^{a_j})\right) & \text{if } p < w(T), \\ \chi_j(p^{a_j}) & \text{if } p \geqslant w(T). \end{cases}$$

Here, we have decomposed $\chi_j = \prod_p \chi_{j,p}$ into characters $\chi_{j,p}$ modulo $p^{v_p(\widetilde{W}(T))}$. If $p \geqslant w(T)$, the factor at p may be written in the more succinct form:

$$\beta_p(\chi_1, \dots, \chi_r) = \prod_{j=1}^{r} \left(1 + \frac{|h_i(p)|}{p}\right)^{-1} \left(1 + \frac{h_i(p)\chi_i(p)}{p}\right) + O_r(H^2 p^{-2}).$$

When making additional assumptions on the functions h_i, the right hand side of (1.2) can be significantly simplified. In many cases the quantities $S_{h_i}(x; q, a)$ are determined by a product of local densities whenever $(a, q) = 1$ and when q is small, that is $q \leqslant (\log x)^{C'}$ for any fixed constant C'. Given such a local-to-global principle for the $S_{h_i}(x; q, a)$, it should also be possible to reinterpret the right hand side of (1.2) as a product of local densities. This is, for instance, the case when each h_i is $\chi_i(n) n^{it_i}$-pretentious for some character χ_i and some $t_i \in \mathbb{R}$. In this case we have $\#\mathscr{E}_i = 1$ in Theorem 1.3, as may be deduced from [1, Theorem 2.1 and Lemma 3.1].

To further illustrate the above, we consider the example of non-negative functions h_i whose values at primes are fairly equidistributed in arithmetic progressions $a \pmod q$ with $\gcd(a, q) = 1$. The following is an immediate corollary to Theorem 1.2.

Corollary 1.4. *Let $h_1, \dots, h_r \in \mathscr{M}$ be non-negative multiplicative functions and suppose that, given any positive constant C', each of the functions h_i satisfies*

$$S_{h_i}(x; q, a) \asymp \frac{q}{\phi(q)} \frac{1}{x \log x} \prod_{\substack{p \leqslant x \\ p \nmid q}} \left(1 + \frac{h_i(p)}{p}\right)$$

for all progressions $a \pmod q$ *with* $q \leqslant (\log x)^{C'}$ *and* $(a,q)=1$. *Assuming the conditions of Theorem 1.2, we then have*

$$\sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_{i=1}^{r} h_i(\varphi_i(\mathbf{n})) \asymp \beta_\infty \prod_p \beta_p, \qquad (T\to\infty),$$

where

$$\beta_\infty = \mathrm{vol}(\mathfrak{K}) \frac{T^s}{(\log T)^r} \prod_{j=1}^{r} \prod_{p\leqslant T} \left(1+\frac{h_i(p)}{p}\right)$$

and

$$\beta_p = \lim_{m\to\infty} \frac{1}{p^{ms}} \sum_{\mathbf{a}\in\mathbb{N}_0} \sum_{\substack{\mathbf{v}\in(\mathbb{Z}/p^m\mathbb{Z})^s\\ v_p(\varphi_i(\mathbf{v}))=a_i}} \prod_{j=1}^{r} h_j(p^{a_j}) \left(1+\frac{h_j(p)}{p}\right)^{-1}.$$

The β_p *satisfy the asymptotic*

$$\beta_p = 1 + O(p^{-2}).$$

Our proof of Theorem 1.2 proceeds via Green and Tao's nilpotent Hardy–Littlewood method (see [10]). This method consists of two main parts, one being to establish that a W-tricked version of any $h \in \mathscr{M}$ is orthogonal to nilsequences, and the second being to show that this W-tricked version of h has a majorant function which is pseudo-random in the sense of [10] and of the 'correct' average order in the sense that its average order is bounded by the average order of the corresponding W-tricked version of $|h|$. For the first part we rely on [15], which establishes the required condition for all $h \in \mathscr{M}$ which satisfy a certain major arc condition. We will show in Section 8 that this major arc condition is implied by Definition 1.1 (iv). The second part, that is the construction of correct-order pseudo-random majorants will take up a large part of this work.

We end this introduction by discussing the implicit condition that $|S_h(x)| \asymp S_{|h|}(x)$, which appears in Theorem 1.2. To start with, we note that Shiu's [17, Theorem 1] applies to $|h|$ for every $h \in \mathscr{M}$. Thus, if $A \in (\mathbb{Z}/q\mathbb{Z})^*$, then

$$S_h(x;q,A) \leqslant S_{|h|}(x;q,A) \ll \frac{q}{\phi(q)} \frac{1}{\log x} \exp\left(\sum_{\substack{p\leqslant x\\ p\nmid q}} \frac{|h(p)|}{p}\right), \quad (x\to\infty), \tag{1.3}$$

uniformly in A and q, provided that $q \leqslant x^{1/2}$. It will follow from Proposition 2.1 in the next section that the main term in Theorem 1.2 only dominates if, for $q = \widetilde{W}(x)$ and $h \in \{h_1,\dots,h_r\}$, the upper bound (1.3) on $S_h(x;q,A)$ is of the correct order for a positive proportion of residues $A \in (\mathbb{Z}/q\mathbb{Z})^*$. Let h' be defined as $h'(n) = h(n)\mathbf{1}_{\gcd(n,W(x))=1}$. Then the latter condition certainly holds if

$$|S_{h'}(x)| \asymp S_{|h'|}(x) \asymp \frac{1}{\log x} \exp\left(\sum_{p\leqslant x} \frac{|h'(p)|}{p}\right).$$

In this regard, recent work of Elliott shows the following.

Lemma 1.5 (Elliott, Elliott–Kish)**.** *Suppose h is a multiplicative function that satisfies conditions (i) and (iii) from Definition 1.1 and define $h' : n \mapsto h(n)\mathbf{1}_{\gcd(n,W(x))=1}$.*

If $f \in \{h, h'\}$, then

$$S_{|f|}(x) \asymp \frac{1}{\log x} \exp\left(\sum_{p \leqslant x} \frac{|f(p)|}{p}\right),$$

and $|S_f(x)| \asymp S_{|f|}(x)$ holds if and only if there exists $t_h \in \mathbb{R}$ such that

$$\sum_{p \text{ prime}} \frac{|h(p)| - \Re(h(p)p^{it_h})}{p} < \infty. \tag{1.4}$$

More precisely, if there exists t_h as above, then

$$S_f(x) = S_{|f|}(x) \frac{x^{-it_h}}{1 - it_h} \prod_{p \leqslant x} \left(\frac{1 + f(p)p^{it_h - 1} + \dots}{1 + |f(p)|p^{-1} + \dots}\right) + o(S_{|f|}(x)). \tag{1.5}$$

Proof. This lemma is a direct application of Elliott [4, Theorem 4] and either Elliott [4, Theorem 2] or Elliott–Kish [5, Lemma 21]. Thus, we only need to check that f satisfies the conditions of these results. To start with, since $|h(n)| \ll_\varepsilon n^\varepsilon$, the sum $\sum |f(q)|/q$ over all proper prime powers $q = p^k$, $k \geqslant 2$, converges. Taking Definition 1.1 (i) into account, we also see that $x^{-1} \sum_{p^k < x} |f(p^k)| \log p^k \ll H$ for $x \geqslant 2$. The condition of [5, Lemma 21] is identical to our condition (iii), the weaker condition from [4, Theorem 2] follows by partial summation. By [5, Lemma 21] (or [4, Theorem 2]), Shiu's bound and condition (iii) we thus have

$$\sum_{\substack{n \leqslant x \\ \gcd(n,W(x))=1}} |h(n)| \asymp \frac{x}{\log x} \prod_{w(x) < p \leqslant x} \left(1 + \frac{|h(p)|}{p}\right) \gg_\varepsilon x(\log x)^{\alpha_h - \varepsilon - 1}. \tag{1.6}$$

□

Related work and an open question. We wish to draw the reader's attention to some related work in the context of bounded pretentious multiplicative functions. A bounded multiplicative function $h : \mathbb{N} \to \mathbb{C}$ is said to be pretentious, if there exists $t_h \in \mathbb{R}$ and a character χ_h such that

$$\sum_{p \text{ prime}} \frac{1 - \Re(h(p)\chi_h(p)p^{it_h})}{p} < \infty. \tag{1.7}$$

The difference between this condition and (1.4) lies in the constant 1 which replaces $|h(p)|$, making (1.7) a somewhat stronger condition. The characteristic function of sums of two squares is one example of a function that is not pretentious because it is 0 at too many primes. For pretentious h it is known (see [1, 9, 12]) that the quantities $S_h(x; q, a)$ are determined by a product of local densities. Klurman [12] recently succeeded in asymptotically evaluating correlations of the form $\sum_{n \leqslant x} f_1(P_1(n)) \dots f_r(P_r(n))$ for bounded pretentious multiplicative functions $f_1, \dots, f_r$ and for arbitrary polynomials $P_1, \dots, P_r \in \mathbb{Z}[x]$.

Frantzikinakis and Host [7] established a Green–Tao theorem, like Theorem 1.2, for pretentious multiplicative functions. In [7], the multiplicative functions h_i are allowed to be complex-valued, they must all be bounded in absolute value by 1 and, in the case where the asymptotic formula corresponding to (1.2) carries a main term, they have the property that $|S_{h_i}(x)| \asymp 1$. Our result, in contrast, only applies to real-valued h_i but it yields an asymptotic with main term so long as (1.4) applies. This includes, for instance, the function $n \mapsto 2^{-\omega(n)}$, where $\omega(n)$ denotes the number of distinct prime factors, or sparse characteristic functions like that of the set of sums of two squares. While the present paper is mainly concerned with the construction of correct-order pseudo-random majorants, no such construction is required in the case of [7] as the trivial majorant given by the all-one function $\mathbf{1}$ is a pseudo-random majorant of the correct average order for every pretentious multiplicative function.

It is an interesting question to investigate if the material in Section 8 can be extended to complex-valued functions that satisfy Elliott's condition (1.4). An affirmative answer would allow one to extend the class $\mathscr{M}$ to which Theorem 1.2 applies to all complex-valued functions that satisfy conditions (i)–(iii), a modified version of (iv), as well as (1.4). This would yield a natural generalisation of both Theorem 1.2 and the result of Frantzikinakis and Host.

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2. Reduction of the main result to a W-tricked version

We begin the proof of Theorem 1.2 by a reduction to the following special case which works in subprogressions whose common difference is a large $w(T)$-smooth integer. This removes potential irregularities that occur when working in progressions modulo small q.

Proposition 2.1. *Let $h_1, \dots, h_r \in \mathscr{M}$ be multiplicative functions. Let $T > 1$ be an integer parameter and let $\varphi_1, \dots, \varphi_r \in \mathbb{Z}[X_1, \dots X_s]$ be linear polynomials as in Theorem 1.2. Given any constant $C > 0$, there exist positive constants B_1 and $B_2 > C$ and, for each T, a $w(T)$-smooth integer $\widetilde{W}(T) \leqslant (\log T)^{B_1}$ divisible by $W(T) = \prod_{p \leqslant w(T)} p$ such that the following holds.*

Let $W_1, \dots, W_r \in [1, (\log T)^{B_1+B_2}]$ be $w(T)$-smooth integers, each divisible by $\widetilde{W}(T)$, and let $W' = \mathrm{lcm}(W_1, \dots, W_r)$. Let $A_1, \dots, A_r$ are integers co-prime to $W(T)$. And, finally, let $\mathfrak{K} \subset [-1,1]^s$ be a convex body with $\mathrm{vol}(\mathfrak{K}) \gg 1$ that is such that $W_i \varphi_i(\frac{T}{W}\mathfrak{K}) + A_i \subset [1, T]$ for each $1 \leqslant i \leqslant r$. Then, as $T \to \infty$, we have

$$\sum_{\mathbf{n} \in \mathbb{Z}^s \cap (T/W')\mathfrak{K}} \prod_{i=1}^{r} h_i(W_i \varphi_i(\mathbf{n}) + A_i) =$$

$$\mathrm{vol}(\mathfrak{K}) \frac{T^s}{W'^s} \prod_{i=1}^{r} S_{h_i}\left(T; \widetilde{W}(T), A_i\right) + o\left(\frac{T^s}{W'^s} \prod_{i=1}^{r} \frac{\log w(T)}{\log T} \prod_{w(T) < p \leqslant T} \left(1 + \frac{|h_i(p)|}{p}\right) \right).$$

The fact that the upper bounds on the W_i and $\widetilde{W}(T)$ take the shape of a fixed power of $\log T$ is essential for a later application of results from [15] which will imply that the function $n \mapsto h_i(W_i n+A_i)-S_{h_i}(T;W_i,A_i)$ is orthogonal to nilsequences. In order to deduce Theorem 1.2 from Proposition 2.1, we will therefore need to truncate certain summations that occur on the way and show that the terms we remove make a negligible contribution. For this purpose we introduce the following exceptional set.

Definition 2.2 (Exceptional set)**.** Let $C > 0$, $T > 1$ and let $\mathscr{S'}_C(T)$ denote the set all positive integers less than T that are divisible by the square of an integer $d > (\log T)^C$.

To justify that $\mathscr{S'}_C(T) \subset \{1,\dots,T\}$ is an exceptional set, let $\varphi_1,\dots,\varphi_r \in \mathbb{Z}[X_1,\dots,X_s]$ be as in Theorem 1.2, let $T > 1$ and $C > 0$. Then, first of all:

$$\sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \sum_{i=1}^{r} \mathbf{1}_{\varphi_i(\mathbf{n})\in\mathscr{S'}_C(T)} \ll \sum_{d>(\log T)^C} \frac{|\mathbb{Z}^s\cap T\mathfrak{K}|}{d^2} \ll |\mathbb{Z}^s\cap T\mathfrak{K}|(\log T)^{-C}. \tag{2.1}$$

If $\alpha = \min(\alpha_{h_1},\dots,\alpha_{h_r})$, then $\sum_{n\leqslant T} h_i(n) \gg T(\log T)^{-1+\alpha}$ for all $1 \leqslant i \leqslant r$. Thus it follows from (2.1) and [3, Lemma 7.9], combined with an application of the Cauchy-Schwarz inequality, that

$$\sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_i h_i(\varphi_i(\mathbf{n}))\mathbf{1}_{\varphi_i(\mathbf{n})\notin\mathscr{S'}_C(T)} = (1+o(1)) \sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_i h_i(\varphi_i(\mathbf{n})), \tag{2.2}$$

provided C is sufficiently large with respect to r, s, the coefficients of the linear forms $\varphi_i - \varphi_i(\mathbf{0})$, and the respective values of H from Definition 1.1 (i) for the h_i. Hence, $\mathscr{S'}_C(T)$ is indeed exceptional.

The following observation is crucial for employing the exceptional set $\mathscr{S'}_C(T)$ in order to later discard all cases in which Proposition 2.1 would need to be applied with a value of $W > (\log T)^C$.

Lemma 2.3. *Let $C > 1$ and suppose that T is sufficiently large. Then every integer $w > (\log T)^{3C}$ that is composed only out of primes $p \leqslant w(T)$ has a square divisor larger than $(\log T)^{2C}$ and, hence, belongs to $\mathscr{S'}_C(T)$.*

Proof. Since $W(T) = \prod_{p\leqslant w(T)} p = \exp\sum_{p\leqslant \log\log T} \log p \sim \log T$. Thus, if T is sufficiently large, then $W(T) < (\log T)^C$. Factorising w as $w_1 w_2^2$ for a square-free integer w_1, it follows that $w_1 \leqslant W(T) < (\log T)^C$. Hence, $w_2^2 > (\log T)^C$. □

Proof of Theorem 1.2 assuming Proposition 2.1. Let $C > 1$ be sufficiently large for (2.2) to hold, let $B_2 > 3C$ be the constant from the statement of Proposition 2.1 with C replaced by $3C$, and define the set

$$\mathscr{W}(T) = \{w \in [1,(\log T)^{B_2}] : p|w \Rightarrow p \leqslant w(T)\}$$

which, by Lemma 2.3, contains all unexceptional $w(T)$-smooth integers.

For any given $T > 1$ and $w \in \mathscr{W}(T)$, let us temporarily write $w\|^T n$ to indicate that $w|n$ but $\gcd(W(T),\frac{n}{w}) = 1$, in order to make the following definition: Given any collection $w_1,\dots,w_r$ of positive integers, we define the set of r-tuples

$$\mathscr{U}(w_1,\dots,w_r) = \left\{ \Big(\varphi_i(\mathbf{v}) - w\widetilde{W}(T)\varphi_i(0)\Big)_{1\leqslant i\leqslant r} : \begin{array}{c} \mathbf{v} \in \{0,\dots w\widetilde{W}(T)-1\}^r \\ w_i\|^T \varphi_i(\mathbf{v}), 1\leqslant i\leqslant r \end{array} \right\},$$

where $w = \mathrm{lcm}(w_1, \ldots, w_r)$. This definition is made in view of the decomposition

$$\varphi_i(M\mathbf{m} + \mathbf{v}) = M\varphi_i(\mathbf{m}) - M\varphi_i(\mathbf{0}) + \varphi_i(\mathbf{v})$$

valid for any $\mathbf{v} \in \mathbb{Z}^r$ and $M \in \mathbb{Z}$.

By (2.2) and Lemma 2.3, the expression from Theorem 1.2 then satisfies

$$\begin{aligned}
&\sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_{i=1}^{r} h_i(\varphi_i(\mathbf{n})) \\
&= (1+o(1)) \sum_{\substack{w_1,\ldots,w_r \\ \in\mathscr{W}(T)}} \sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_{i=1}^{r} h_i(\varphi_i(\mathbf{n}))\mathbf{1}_{w_i \|^T \varphi_i(\mathbf{n})} \\
&= (1+o(1)) \sum_{\substack{w_1,\ldots,w_r \\ \in\mathscr{W}(T)}} \sum_{\substack{(U_1,\ldots,U_r) \\ \in\mathscr{U}(w_1,\ldots,w_r)}} \sum_{\substack{\mathbf{m}:\, w\widetilde{W}(T)\mathbf{m} \\ \in\mathbb{Z}^s\cap T\mathfrak{K}}} \prod_{i=1}^{r} h_i\left(w\widetilde{W}(T)\varphi_i(\mathbf{m}) + U_i\right),
\end{aligned}$$

where, again, $w = \mathrm{lcm}(w_1, \ldots, w_r)$. Note that we made use of the fact that the $\mathfrak{K}$ is convex in order to restrict the summation over $\mathbf{m}$ to the set $\{\mathbf{m} \in \mathbb{Z}^s : w\widetilde{W}(T)\mathbf{m} \in \mathbb{Z}^s \cap T\mathfrak{K}\}$.

Invoking the multiplicativity of the h_i, the above becomes

$$(1+o(1)) \sum_{\substack{w_1,\ldots,w_r \\ \in\mathscr{W}(T)}} \left(\prod_{j=1}^{r} h_j(w_j)\right) \sum_{\substack{(U_1,\ldots,U_r) \\ \in\mathscr{U}(w_1,\ldots,w_r)}} \sum_{\substack{\mathbf{m}\in \\ \mathbb{Z}^s\cap(T/w\widetilde{W}(T))\mathfrak{K}}} \prod_{i=1}^{r} h_i\left(\frac{w\widetilde{W}(T)}{w_i}\varphi_i(\mathbf{m}) + \frac{U_i}{w_i}\right).$$

Since $\gcd(W(T), U_i/w_i) = 1$ for all i, Proposition 2.1 applies to the inner summation over $\mathbf{m}$ and we deduce that the above equals

$$\begin{aligned}
&(1+o(1)) \sum_{\substack{w_1,\ldots,w_r \\ \in\mathscr{W}(T)}} \left(\prod_{j=1}^{r} h_j(w_j)\right) \\
&\sum_{\substack{(U_1,\ldots,U_r) \\ \in\mathscr{U}(w_1,\ldots,w_r)}} \frac{T^s}{(w\widetilde{W}(T))^s} \left(\mathrm{vol}(\mathfrak{K}) \prod_{i=1}^{r} S_{h_i}\left(T; \widetilde{W}(T), \frac{U_i}{w_i}\right) + \prod_{i=1}^{r} S_{|h_i|}\left(T; \widetilde{W}(T), \frac{U_i}{w_i}\right)\right).
\end{aligned}$$

Finally, by invoking Shiu's bound, we obtain

$$\begin{aligned}
&\sum_{\substack{w_1,\ldots,w_r \\ \in\mathscr{W}(T)}} \sum_{\substack{A_1,\ldots,A_r \\ \in(\mathbb{Z}/\widetilde{W}\mathbb{Z})^*}} \left(\prod_{i=1}^{r} h_i(w_i) S_{h_i}\Big(T; \widetilde{W}, A_i\Big)\right) \frac{\mathrm{vol}(\mathfrak{K})T^s}{(w\widetilde{W})^s} \sum_{\substack{\mathbf{v}\in \\ (\mathbb{Z}/w\widetilde{W}\mathbb{Z})^s}} \prod_{j=1}^{r} \mathbf{1}_{\varphi_j(\mathbf{v})\equiv w_j A_j \pmod{w_j\widetilde{W}}} \\
&\qquad + o\left(\frac{T^{r+s}}{(\log T)^r} \prod_{j=1}^{r} \prod_{p\leqslant T} \left(1 + \frac{|h_i(p)|}{p}\right)\right),
\end{aligned}$$

which completes the proof. □

3. Majorants for multiplicative functions

For any function $h \in \mathscr{M}$, we obtain a simple majorant function $h' : \mathbb{N} \to \mathbb{R}$ by setting $h'(n) := h^\sharp(n)h^\flat(n)$, where $h^\sharp$ and $h^\flat$ are multiplicatively defined by

$$h^\sharp(p^k) = \max(1, |h(p)|, \dots, |h(p^k)|)$$

and

$$h^\flat(p^k) = \min(1, |h(p^k)|),$$

respectively. It is immediate that $|h(n)| \leqslant h'(n)$ for all $n \in \mathbb{N}$. The function $h^\sharp$ belongs to the class of functions for which pseudo-random majorants were already constructed in [3, §7]. Our pseudo-random majorant for h will arise as a product of separate majorants for $h^\sharp$ and $h^\flat$. Thus, our main task here is to construct general pseudo-random majorants for bounded multiplicative functions.

4. A majorant for $h^\sharp$

Before we turn to the case of bounded multiplicative functions, let us record the known pseudo-random majorant for $h^\sharp$. For this purpose, set $g = \mu * h^\sharp$ and define for any $\gamma \in (0, 1/2)$ the truncation

$$h_\gamma^{(T)}(m) = \sum_{d \in \mathbb{N}} \mathbf{1}_{d|m} g(d) \chi\Big(\frac{\log d}{\log T^\gamma}\Big)$$

of the convolution $h^\sharp = \mathbf{1} * g$, where $\chi : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ is a smooth function with support in $[-1, 1]$, which is monoton on $[-1, 0]$ and $[0, 1]$ and has the property that $\chi(x) = 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$. For any fixed value of $\gamma \in (0, 1/2)$, [3, Proposition 7.6] then shows that

$$h^\sharp(n) \ll \nu_\sharp(n)$$

for $n \leqslant T$, where

$$\nu_\sharp(m) = \sum_{\kappa=4/\gamma}^{[(\log\log T)^3]} \sum_{\lambda=\lceil \log_2 \kappa - 2 \rceil}^{[\log_2((\log\log T)^3)]} \sum_{u \in U(\lambda,\kappa)} H^\kappa \mathbf{1}_{u|m} h^\sharp(u) h_\gamma^{(T)} \left(\frac{m}{\prod_{p|u} p^{v_p(m)}} \right) + \mathbf{1}_{m \in \mathscr{S}} h^\sharp(m),$$

with an exceptional set

$$\mathscr{S} = \left\{ n \leqslant T : \Big(\exists p. v_p(n) \geqslant \max\{2, C_1 \log_p \log T\}\Big) \text{ or } \prod_{p \leqslant T^{1/(\log\log T)^3}} p^{v_p(m)} \geqslant T^{\gamma/\log\log T} \right\},$$

and where each set $U(\lambda, \kappa)$ is a sparse subset of the integers up to N^γ that is defined as follows. Set $\omega(\lambda, \kappa) = \left\lceil \frac{\gamma\kappa(\lambda+3-\log_2 \kappa)}{200} \right\rceil$ and $I_\lambda = [T^{1/2^{\lambda+1}}, T^{1/2^\lambda}]$. Then

$$U(\lambda, \kappa) = \begin{cases} \{1\}, & \text{if } \kappa = 4/\gamma \text{ and } \lambda = \log_2 \kappa - 2, \\ \emptyset, & \text{if } \kappa = 4/\gamma \text{ and } \lambda \neq \log_2 \kappa - 2, \\ \left\{ p_1 \dots p_{\omega(\lambda,\kappa)} : \begin{array}{l} p_i \in I_\lambda \text{ distinct primes} \\ h^\sharp(p_i) \neq 1 \end{array} \right\}, & \text{if } \kappa > 4/\gamma. \end{cases}$$

We note for later reference that $p \geqslant T^{1/(\log\log T)^3}$ for any prime divisor $p|u$ of any $u \in U(\lambda, \kappa)$ that appears in the definition of $\nu_\sharp$.

5. Majorants for bounded multiplicative functions

Any bounded multiplicative function $h^\flat$ has the property that whenever $n'|n$ is a divisor that is coprime to its cofactor n/n', then $h^\flat(n) \leqslant h^\flat(n')$. The key step in turning this simple observation into the construction of a pseudo-random majorant is to find a systematic way of assigning to any integer n a suitable divisor n'. The main property this map must have is that the pre-image of any divisor n' should be easily reconstructible, a property which will allow us to swap the order of summation in later computations. The following assignment already featured in Erdős's work [6] on the divisor function:

Given a cut-off parameter x and an integer $n \in [x^\gamma, x]$, let $D_\gamma(n)$ denote the largest divisor of n that is of the form

$$\prod_{p \leqslant Q} p^{v_p(n)}, \quad (Q \in \mathbb{N})$$

but does not exceed x^γ. If $m \in (0, x^\gamma)$ is an integer then its inverse image takes the form

$$D_\gamma^{-1}(m) = \{\delta m \in [x^\gamma, x] : P^+(m) < P^-(\delta)\},$$

where $P^+(m)$, resp. $P^-(m)$, denote the largest, resp. smallest, prime factor of m. For our purpose it turns out to be of advantage to restrict attention to divisors

$$m \in \langle \mathscr{P}_\flat \rangle = \{m : p|m \implies p \in \mathscr{P}_\flat\},$$

where

$$\mathscr{P}_\flat = \{p : h(p) < 1\}.$$

Thus, if $n \in [1, x]$ is an integer that factorises as $n = mm'$ with $m \in \langle \mathscr{P}_\flat \rangle$ and $m' \notin \langle \mathscr{P}_\flat \rangle$, then we set

$$D'_\gamma(n) = \begin{cases} m & \text{if } m \leqslant x^\gamma \\ D_\gamma(m) & \text{if } m > x^\gamma \end{cases}.$$

Our next aim is to show that a sufficiently smoothed version of the function D'_γ can be written as a truncated divisor sum. To detect whether a given divisor $\delta|n$ is of the form $\prod_{p \leqslant Q} p^{v_p(n)}$ for some Q or, equivalently, whether $m = \frac{n}{\delta}$ has no prime factor $p \leqslant Q$, we make use of a sieve majorant similar to the one considered in [10, Appendix D]. The essential differences are that the parameter corresponding to Q cannot be fixed in our application and that the divisor sum will be restricted to elements of the set $\langle \mathscr{P}_\flat \rangle$. Thus, let $\sigma_\flat : \mathbb{R} \times \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be defined as

$$\sigma_\flat(Q; m) = \left(\sum_{\substack{d|m \\ d \in \langle \mathscr{P}_\flat \rangle}} \mu(d) \chi\left(\frac{\log d}{\log Q}\right) \right)^2,$$

where $\chi : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ is a smooth function with support in $[-1,1]$ and the property $\chi(x) = 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$. This yields a non-negative function with the property that $\sigma_\flat(Q;m) = 1$ if m is free from prime factors $p \in \mathscr{P}_\flat$ with $p \leqslant Q$. Setting, for $1 \leqslant n \leqslant x$,

$$\nu'(n) = \sum_{\substack{m|n \\ m \in \langle \mathscr{P}_\flat \rangle \\ m < x^\gamma}} h^\flat(m) \sigma_\flat\left(x^\gamma; \frac{n}{m}\right) + \sum_{\substack{Q \leqslant x^\gamma \\ Q \in \mathscr{P}_\flat}} \sum_{\substack{m|n \\ p|m \Rightarrow p<Q}} \sum_{Q^k|n} h^\flat(mQ^k) \mathbf{1}_{m<x^\gamma} \mathbf{1}_{Q^k m \geqslant x^\gamma} \sigma_\flat\left(Q; \frac{n}{mQ^k}\right),$$

we obtain a (preliminary) majorant $\nu' : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ for $h^\flat$. The first of two small modifications consists of inserting smooth cut-offs for m and $Q^k m$, leading to the majorant function $\nu'' : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ defined as

$$\nu''(n) = \sum_{\substack{m|n \\ m \in \langle \mathscr{P}_\flat \rangle}} h^\flat(m) \chi\left(\frac{\log m}{\gamma \log x}\right) \sigma\left(x^\gamma; \frac{n}{m}\right)$$
$$+ \sum_{Q \in \mathscr{P}_\flat} \sum_{\substack{m|n \\ m \in \mathscr{P}_\flat \\ p|m \Rightarrow p<Q}} \sum_{Q^k|n} h^\flat(mQ^k) \lambda\left(\frac{\log Q^k \delta}{\log x}\right) \sigma\left(Q; \frac{n}{\delta Q^k}\right),$$

where λ is a smooth cut-off of the interval $[\gamma, 2\gamma]$ which is supported in $[\gamma/2, 4\gamma]$ and takes the value 1 on the interval $[\gamma, 2\gamma]$. To carry out the second simplification, we exclude a sparse exceptional set related to the one from Section 4. If $Q < T^{\gamma/(\log\log T)^3}$, then an integer $n < T$ belongs to the exceptional set $\mathscr{S}$ from Section 4 if it has a divisor of the form $Q^k m > T^\gamma$, where $p|m \Rightarrow p < Q$. If $Q > T^{\gamma/(\log\log T)^3} = (\log T)^{\gamma(\log T)/(\log\log T)^4}$ and if T is sufficiently large, then $Q^2 > (\log T)^{C_1}$ so that any multiple of Q^2 again also belongs to $\mathscr{S}$. Thus, by defining

$$\nu_\flat(n) = \sum_{\substack{m|n \\ m \in \langle \mathscr{P}_\flat \rangle}} h^\flat(m) \chi\left(\frac{\log m}{\gamma \log x}\right) \sigma\left(x^\gamma; \frac{n}{m}\right)$$
$$+ \sum_{\substack{Q|n \\ Q \text{ prime} \\ Q > x^{\gamma/(\log\log T)^3}}} \sum_{\substack{m|n \\ p|m \Rightarrow p<Q}} h^\flat(Qm) \lambda\left(\frac{\log Qm}{\log x}\right) \sigma\left(Q; \frac{n}{Qm}\right),$$

we obtain a positive function with the property

$$|h_\flat(n)| \ll \nu_\flat(n) + \mathbf{1}_{\mathscr{S}}(n)$$

for any integer $n \in [1, x]$, provided x is sufficiently large with respect to the value of C_1 from the definition of $\mathscr{S}$. By construction (cf. [6] or [13, Lemmas 3.2 and 3.3]) it follows that $\sum_{n \leqslant x} \mathbf{1}_{\mathscr{S}}(n) \ll x(\log x)^{-C_1/2}$. Thus, choosing C_1 sufficiently large, we may in view of (1.6) ensure that

$$\sum_{n \leqslant x} (1 + |h(n)|) \mathbf{1}_{\mathscr{S}}(n) = o(\sum_{n \leqslant x} |h(n)|),$$

i.e., that $\mathscr{S}$ is indeed an exceptional set.

6. The product $\nu^\sharp\nu^\flat$ is pseudo-random

We will now sketch a proof of the linear forms estimate for ν, which by [10, App. D] implies that ν is pseudo-random. This linear forms estimate can only be established after removing the contribution of small prime factors, that is, only when working with a W-trick. Recall that $w(x)$ was defined in Section 1.

Proposition 6.1 (Linear forms estimate)**.** *Let $T > 1$ be an integer parameter and $B \geqslant 1$ a constant. For each T and each $1 \leqslant i \leqslant r$, let $1 \leqslant W_i \leqslant (\log T)^B$ be a $w(T)$-smooth integer that is divisible by $W(T)$ and let $0 \leqslant A_i \ll W_i$ be coprime to $W(T)$. Let $D > 1$ be constant, let $1 \leqslant r, s < D$ be integers and let $\phi_1, \dots, \phi_r \in \mathbb{Z}[X_1, \dots X_s]$ be linear polynomials whose non-constant parts are pairwise independent and have coefficients that are bounded in absolute value by D. The constant coefficients $h_i(0)$ may depend on T provided $h_i(0) = O_D(T)$. Suppose that $\mathfrak{K} \subset [-1,1]^s$ is a convex body with* $\mathrm{vol}(\mathfrak{K}) \gg 1$ *and that $W_i\phi_i(T\mathfrak{K}) + A_i \subset [1, W_i T]$ for each $1 \leqslant i \leqslant r$. Then*

$$\frac{1}{T^s \operatorname{vol}\mathfrak{K}} \sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_{i=1}^{r} \nu_{h_i}(W_i\phi_i(\mathbf{n}) + A_i) = (1+o(1)) \prod_{i=1}^{r} \left(\frac{1}{T} \sum_{n \leqslant T} \nu_{h_i}(W_i n + A_i) \right)$$

as $T \to \infty$, provided γ is sufficiently small.

Proof. Inserting all definition, we obtain

$$\begin{aligned}
&\frac{1}{T^s \operatorname{vol}\mathfrak{K}} \sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_{i=1}^{s} \nu_{h_i}(\phi_i(\mathbf{n})) \qquad (6.1)\\
&=\frac{1}{T^s \operatorname{vol}\mathfrak{K}} \sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_{i=1}^{s} \nu^\sharp_{h_i}(\phi_i(\mathbf{n}))\nu^\flat_{h_i}(\phi_i(\mathbf{n}))\\
&=\frac{1}{T^s \operatorname{vol}\mathfrak{K}} \sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_{j=1}^{r} \Bigg(\sum_{\kappa_i=4/\gamma}^{[(\log\log T)^3]} \sum_{\lambda_i=\lceil \log_2 \kappa_i - 2\rceil}^{[\log_2((\log\log T)^3)]} \sum_{u_j\in U_j(\lambda_j,\kappa_j)} \sum_{\substack{d_j:\\ p\mid d_j \Rightarrow p>w(T)\\ \gcd(d_j,u_j)=1}}\\
&\qquad \sum_{Q_i \text{ prime}} \sum_{\substack{m_i:\\ p\mid m_i\Rightarrow\\ w(T)<p<Q_i}} \sum_{\substack{\delta_j,\delta'_j:\ p\mid\delta_j\delta'_j\\ \Rightarrow p>w(T)}} \mathbf{1}_{\operatorname{lcm}(\delta_j m_j Q_j, \delta'_j m_j Q_j, u_j, d_j)\mid\phi_j(\mathbf{n})}\\
&\qquad H^{\kappa_j} h^\sharp(u_j) g^\sharp_j(d_j)\mu(\delta_j)\mu(\delta'_j)h^\flat(m_j)h^\flat(Q_j)\\
&\qquad \lambda\left(\frac{\log Q_j m_j}{\log x}\right)\chi\left(\frac{\log d_j}{\log x^\gamma}\right)\chi\left(\frac{\log \delta_j}{\log Q_j}\right)\chi\left(\frac{\log \delta'_j}{\log Q_j}\right)\Bigg).
\end{aligned}$$

Let

$$\alpha_{\boldsymbol{\phi}}(p^{c_1},\dots,p^{c_r}) = \frac{1}{p^{ms}} \sum_{\mathbf{u}\in(\mathbb{Z}/p^m\mathbb{Z})^s} \prod_{i=1}^r \mathbf{1}_{p^{c_i}|\phi_i(\mathbf{u})},$$

where $m=\max(c_1,\dots,c_r)$ for any prime p and extend $\alpha_{\boldsymbol{\phi}}$ to composite arguments multiplicatively. Writing

$$\Delta_j = \mathrm{lcm}(\delta_j m_j Q_j, \delta_j' m_j Q_j, u_j, d_j),$$

it follows (cf. [10, Appendix A]) that

$$\frac{1}{x^s \operatorname{vol}\mathfrak{K}} \sum_{\mathbf{n}\in\mathbb{Z}^s\cap x\mathfrak{K}} \prod_{j=1}^r \mathbf{1}_{\Delta_j|\phi_j(\mathbf{n})} = \alpha_{\boldsymbol{\phi}}(\Delta_1,\dots,\Delta_r)\left(1+O\left(\frac{x^{-1+O(\gamma)}}{\operatorname{vol}\mathfrak{K}}\right)\right).$$

Thus, the expression above equals

$$\begin{aligned}\left(1+O\left(\frac{T^{-1+O(\gamma)}}{\operatorname{vol}\mathfrak{K}}\right)\right) &\sum_{\boldsymbol{\kappa},\boldsymbol{\lambda},\mathbf{u}} \sum_{\mathbf{Q},\mathbf{m},\boldsymbol{\delta},\boldsymbol{\delta}',\mathbf{d}} \alpha_{\boldsymbol{\phi}}(\Delta_1,\dots,\Delta_r)\\ &\prod_{j=1}^r H^{\kappa_j} h^\sharp(u_j) g_j^\sharp(d_j)\mu(\delta_j)\mu(\delta_j')h^\flat(m_jQ_j) \qquad (6.2)\\ &\times \lambda\left(\frac{\log Q_j m_j}{\log x}\right)\chi\left(\frac{\log d_j}{\log x^\gamma}\right)\chi\left(\frac{\log \delta_j}{\log Q_j}\right)\chi\left(\frac{\log \delta_j'}{\log Q_j}\right)\end{aligned}$$

with the same summation conditions as above. Our aim is to show that the above expression equals

$$\begin{aligned}\left(1+o(1)+O\left(\frac{T^{-1+O(\gamma)}}{\operatorname{vol}\mathfrak{K}}\right)\right) &\prod_{j=1}^r \sum_{\kappa_j,\lambda_j,u_j} \sum_{Q_j,m_j,\delta_j,\delta_j',d_j} \frac{H^{\kappa_j} h^\sharp(u_j) g_j^\sharp(d_j)\mu(\delta_j)\mu(\delta_j')h^\flat(m_jQ_j)}{\Delta_j}\\ \times \lambda\left(\frac{\log Q_j m_j}{\log x}\right)&\chi\left(\frac{\log d_j}{\log x^\gamma}\right)\chi\left(\frac{\log \delta_j}{\log Q_j}\right)\chi\left(\frac{\log \delta_j'}{\log Q_j}\right), \qquad (6.3)\end{aligned}$$

again with the summation conditions from (6.1) in place. Note that (6.3) no longer features the ϕ_i, which can only be achieved because we are working with a W-trick.

The proof of the above equality follows the approach of [3, §9] closely, although the situation here bears a few extra difficulties. All essential tools in this analysis were derived or developed starting out from material in [10, Appendix D]. The main steps are as follows:

(1) For every fixed choice of $\boldsymbol{\kappa}=(\kappa_1,\dots,\kappa_r)$ and $\boldsymbol{\lambda}=(\lambda_1,\dots\lambda_r)$ satisfying the summation conditions from (6.1), the sum over $\{\mathbf{u} : u_i \in U(\kappa_i,\lambda_i), 1\leqslant i\leqslant r\}$ in (6.2) (resp. (6.3)) can be replaced by $(1+o(1))\sum'_{\mathbf{u}}$, where $\sum'$ indicates that the summation is restricted to tuples $(u_1,\dots,u_r)$ of pairwise coprime integers such that $\gcd(u_i, \prod_{j=1}^r Q_j m_j d_j \delta_j \delta_j') = 1$ for every $1\leqslant i\leqslant r$.

To prove this one shows that the sum over the excluded choices of $(u_1,\dots,u_r)$ makes a negligible contribution. This follows as in the proof of Claim 2 in [14, §9] by means of the

Cauchy-Schwarz inequality by combining the lower bound $p \geqslant T^{(\log\log T)^{-3}}$ on the prime factors of any $u_i \in U(\kappa_i, \lambda_i)$ and the observation that

$$\sum_{\substack{T^{(\log\log T)^{-3}} \\ \leqslant p < T^\gamma}} \sum_{\mathbf{n} \in \mathbb{Z}^s \cap T\mathfrak{K}} \mathbf{1}_{p^2 | \prod_i \phi_i(\mathbf{n})} \ll_r \sum_{\substack{T^{(\log\log T)^{-3}} \\ \leqslant p < T^\gamma}} \frac{|\mathbb{Z}^s \cap T\mathfrak{K}|}{p^2} \ll_r |\mathbb{Z}^s \cap T\mathfrak{K}| \exp\left(-\frac{\log T}{(\log\log T)^3}\right)$$

with the following crude bound on the second moment of (6.2):

$$H^{2r(\log\log T)^3} \sum_{\mathbf{n} \in \mathbb{Z}^s \cap T\mathfrak{K}} \prod_{i=1}^r d(\phi_i(\mathbf{n}))^6 g^\sharp(\phi_i(\mathbf{n})) \ll H^{2r(\log\log T)^3} |\mathbb{Z}^s \cap T\mathfrak{K}| (\log T)^{O_{H,r}(1)},$$

which follows from [3, Proposition 7.9] and where d denotes the divisor function.

(2) Since the primes Q_i satisfy the lower bound $Q_i > T^{(\log\log T)^{-3}}$, we may by proceeding exactly as in step (1), replace the summation $\sum_{\mathbf{Q}}$ by $(1+o(1)) \sum'_{\mathbf{Q}}$, where $\sum'$ indicates that the summation is restricted to pairwise distinct primes $Q_1, \dots, Q_r$ such that in addition $\gcd(Q_i, \prod_{j=1}^r m_j d_j \delta_j \delta'_j) = 1$.

(3) With the restrictions to $\sum'_{\mathbf{u}}$ and $\sum'_{\mathbf{Q}}$ from (1) and (2) in place, we have in the argument of (6.2)

$$\alpha_{\boldsymbol{\phi}}(\Delta_1, \dots, \Delta_r) = \frac{\alpha_{\boldsymbol{\phi}}(\tilde{\Delta}_1, \dots, \tilde{\Delta}_r)}{u_1 \dots u_r Q_1 \dots Q_r}$$

where $\tilde{\Delta}_i = \Delta_i / (Q_i u_i)$ for each i. Observe further that

$$c_0 = \sum_{\boldsymbol{\kappa}} \sum_{\boldsymbol{\lambda}} \sum_{\mathbf{u}} \prod_{j=1}^r \frac{H^{\kappa_j} h^\sharp(u_j)}{u_j} < \infty. \tag{6.4}$$

(4) We replace $\chi(\log m / \log Q)$ and $\lambda(\log m/\log Q))$ by multiplicative functions in m, using the Fourier-type transforms

$$e^x \chi(x) = \int_{\mathbb{R}} \theta(\xi) e^{-ix\xi} d\xi, \quad e^x \lambda(x) = \int_{\mathbb{R}} \theta'(\xi) e^{-ix\xi} d\xi.$$

Setting $I = [-(\log T^\gamma)^{1/2}, (\log T^\gamma)^{1/2}]$, one obtains

$$\chi\left(\frac{\log m}{\log Q}\right) = \int_I m^{-\frac{1+i\xi}{\log Q}} \theta(\xi) d\xi + O_E\left(m^{-\frac{1}{\log Q}} (\log T^\gamma)^{-E}\right)$$

and an analogous expression for λ.

Inserting the new expressions for χ and λ at all instances in (6.2) (resp. (6.3)) and multiplying out, we obtain a main term and an error term. Any integral occurring in the main term runs over the compact interval I and, thanks to the factor $\alpha_{\boldsymbol{\phi}}(\Delta_1, \dots, \Delta_r)$, the summation over $\boldsymbol{\kappa}, \boldsymbol{\lambda}, \mathbf{u}, \mathbf{Q}, \mathbf{m}, \boldsymbol{\delta}, \boldsymbol{\delta}', \mathbf{d}$ is absolutely convergent. Thus, in the main term we can swap sums and integrals. Taking into account steps (1)–(3), (6.2) then becomes

$$
\begin{aligned}
&(1+o(1)) \sum_{\boldsymbol{\kappa}} \sum_{\boldsymbol{\lambda}} \sum_{\mathbf{u}} \prod_{j=1}^{r} \frac{H^{\kappa_j} h^{\sharp}(u_j)}{u_j} \\
&\times \int_I \cdots \int_I \sum_{\mathbf{Q}} \sum_{\mathbf{m},\boldsymbol{\delta},\boldsymbol{\delta}',\mathbf{d}} \frac{\alpha_{\boldsymbol{\phi}}(\tilde{\Delta}_1,\ldots,\tilde{\Delta}_r)}{Q_1,\ldots,Q_r} \left(\prod_{j=1}^{r} J_j^* \ \theta'(\xi_{3,j})\theta(\xi_{4,j})\theta(\xi_{5,j})\theta(\xi_{6,j}) \right) \mathbf{d}\boldsymbol{\xi} \\
&+ O_E \left(\frac{1}{(\log T)^E} \sum_{\mathbf{Q}\mathbf{m},\boldsymbol{\delta},\boldsymbol{\delta}',\mathbf{d}} \left(\prod_{i=1}^{r} \frac{H^{\Omega(d_i)}}{(Q_i m_i d_i \delta_i \delta_i')^{1/\log T^{\gamma}}} \right) \frac{\alpha_{\boldsymbol{\phi}}(\tilde{\Delta}_1,\ldots,\tilde{\Delta}_r)}{Q_1,\ldots,Q_r} \right), \qquad (6.5)
\end{aligned}
$$

where

$$J_j^* = h(Q_j) Q_j^{-\frac{1+i\xi_3}{\gamma \log x}} J_j(Q_j)$$

and

$$J_j(Q_j) = g_j^{\sharp}(d_j) h(m_j) m_j^{-\frac{1+i\xi_3}{\gamma \log x}} d_j^{-\frac{1+i\xi_4}{\gamma \log x}} \mu(\delta_j)\mu(\delta_j') \delta_j^{-\frac{1+i\xi_5}{\log Q_j}} {\delta'}_j^{-\frac{1+i\xi_6}{\log Q_j}}.$$

To proceed further, the following two lemmas are required

Lemma 6.2. *Suppose $\boldsymbol{\kappa}, \boldsymbol{\lambda}$ and $u_i \in U(\kappa_i, \lambda_i)$ for $1 \leqslant i \leqslant r$ are fixed. Then we have*

$$\sum_{\mathbf{m},\boldsymbol{\delta},\boldsymbol{\delta}',\mathbf{d}} \alpha_{\boldsymbol{\phi}}(\tilde{\Delta}_1,\ldots,\tilde{\Delta}_r) \left(\prod_{j=1}^{r} J_j(Q_j) \right) = (1+o(1)) \prod_{j=1}^{r} \sum_{m_j,\delta_j,\delta_j',d_j} \frac{J_j(Q_j)}{\tilde{\Delta}_j}$$

for each choice of primes $Q_1,\ldots,Q_r \geqslant T^{(\log\log T)^{-3}}$.

Lemma 6.3. *For each $1 \leqslant j \leqslant r$, we have*

$$
\begin{aligned}
&\int_I \cdots \int_I \left| \sum_{Q_j,m_j,\delta_j,\delta_j',d_j} \frac{J_j^*}{Q_j \tilde{\Delta}_j} \theta'(\xi_{3,j})\theta(\xi_{4,j})\theta(\xi_{5,j})\theta(\xi_{6,j}) \right| \mathbf{d}\boldsymbol{\xi}_j \\
&\ll \frac{\log w(x)}{\log x} \prod_{w(x)<p<x^{\gamma}} \left(1 + \frac{|h(p)|}{p} \right).
\end{aligned}
$$

The two lemmas above correspond to [3, Lemmas 9.4 and 9.3]. The proof of Lemma 6.2 is very similar to that of [3, Lemmas 9.4] rather fairly lengthy so that we omit it here. The proof of Lemma 6.3, in contrast, needs to work with significantly weaker assumptions on the Dirichlet series involved than the corresponding one from [3]. We will therefore carry out this proof below. Before we do so, let us, however, show how to deduce the equality of (6.2) and (6.3) and, thus, complete the proof of Proposition 6.1.

By applying first Lemma 6.2 and then Lemma 6.3, we obtain

$$\int_I \cdots \int_I \sum_{\mathbf{Q}} \bigg(\sum_{\mathbf{m},\boldsymbol{\delta},\boldsymbol{\delta}',\mathbf{d}} \frac{\alpha_{\boldsymbol{\phi}}(\tilde{\Delta}_1, \dots, \tilde{\Delta}_r)}{Q_1, \dots, Q_r} \prod_{j=1}^r J_j^* \\ - \prod_{j=1}^r h(Q_j) Q_j^{-1-\frac{1+i\xi_3}{\gamma \log x}} \sum_{m_j,\delta_j,\delta_j',d_j} \frac{J_j(Q_j)}{\tilde{\Delta}_j} \bigg) \prod_{j'=1}^r \theta'(\xi_{3,j'}) \dots \theta(\xi_{6,j'}) \mathbf{d}\boldsymbol{\xi}$$

$$= o\Bigg(\int_I \bigg| \prod_j \sum_{Q_j,m_j,\delta_j,\delta_j',d_j} \frac{J_j^*}{Q_j \tilde{\Delta}_j} \theta'(\xi_{3,j}) \dots \theta(\xi_{6,j}) \bigg| \mathrm{d}\boldsymbol{\xi} \Bigg)$$

$$= o\left(\frac{\log w(x)}{\log x} \prod_{w(x)<p<x^\gamma} \left(1 + \frac{h(p)}{p}\right) \right).$$

This completes the the proof of Proposition 6.1, assuming Lemma 6.3. □

It remains to prove Lemma 6.3. We emphasise that, although this lemma corresponds to [3, Lemma 9.3], the proof below is significantly stronger in that it does not rely on any assumption on the behaviour close to $s = 1$ of the Dirichlet series attached to $h^\sharp$ or to $h^\flat$.

Proof of Lemma 6.3. We proceed by decomposing the sum in the integrand into Euler products, keeping in mind that the contribution of higher prime powers only affects the implied constant. We have

$$\left| \sum_{Q_j,m_j,\delta_j,\delta_j',d_j} \frac{J_j^*}{Q_j m_j d_j \operatorname{lcm}(\delta_j, \delta_j')} \right|$$

$$\ll \left| \prod_{\substack{p'>w(x) \\ p'\notin \mathscr{P}_\flat}} \left(1 + \frac{g(p')}{p'^{1+\frac{1+i\xi_4}{\gamma \log x}}}\right) \right| \times$$

$$\Bigg\{ \left| \prod_{\substack{p>w(x) \\ p\in \mathscr{P}_\flat}} \left(1 + \frac{h^\flat(p)}{p^{1+\frac{1+i\xi_3}{\gamma \log x}}} - \frac{1}{p^{1+\frac{1+i\xi_5}{\gamma \log x}}} - \frac{1}{p^{1+\frac{1+i\xi_6}{\gamma \log x}}} + \frac{1}{p^{1+\frac{2+i\xi_5+i\xi_6}{\gamma \log x}}}\right) \right|$$

$$+ \left| \sum_{Q\in \mathscr{P}_\flat} \frac{h^\flat(Q)}{Q^{1+\frac{1+i\xi_3}{\gamma \log x}}} \prod_{\substack{p>w(x) \\ p\in \mathscr{P}_\flat}} \left(1 + \frac{h^\flat(p)\mathbf{1}_{p<Q}}{p^{1+\frac{1+i\xi_3}{\gamma \log x}}} - \frac{1}{p^{1+\frac{1+i\xi_5}{\log Q}}} - \frac{1}{p^{1+\frac{1+i\xi_6}{\log Q}}} + \frac{1}{p^{1+\frac{2+i\xi_5+i\xi_6}{\log Q}}}\right) \right| \Bigg\}.$$

After splitting the above factors of the form $(1 + \frac{a_{1,p}}{p^s} + \dots + \frac{a_{4,p}}{p^s})$ into individual factors $(1 + \frac{a_{i,p}}{p^s})$, we make use of the following bounds to analyse the products $\prod_p (1 + \frac{a_{i,p}}{p^s})$ with negative coefficients $a_{i,p} = -1$. We have

$$\begin{aligned}
&\Bigg| \prod_{\substack{p>w(x)\\ p\in\mathscr{P}_\flat}} \left(1 - \frac{1}{p^{1+\frac{1+i\xi}{\gamma\log x}}}\right) \Bigg| \\
&\qquad\ll \left| \zeta^{-1}\left(1+\frac{1+i\xi}{\gamma\log x}\right) \prod_{\substack{p>w(x)\\ p\notin\mathscr{P}_\flat}} \left(1 + \frac{1}{p^{1+\frac{1+i\xi}{\gamma\log x}}}\right) \prod_{q<w(x)} \left(1 + \frac{1}{q^{1+\frac{1+i\xi}{\gamma\log x}}}\right) \right| \\
&\qquad\ll \left|\frac{1+i\xi}{\gamma\log x}\right| \prod_{\substack{p>w(x)\\ p\notin\mathscr{P}_\flat}} \left(1 + \frac{1}{p^{1+\frac{1}{\gamma\log x}}}\right) \prod_{q<w(x)} \left(1+\frac{1}{q}\right) \\
&\qquad\ll \left|\frac{1+i\xi}{\gamma\log x}\right| \prod_{\substack{w(x)<p<x^\gamma\\ p\notin\mathscr{P}_\flat}} \left(1+\frac{1}{p}\right) \prod_{q<w(x)} \left(1+\frac{1}{q}\right) \\
&\qquad\ll |1+i\xi| \prod_{\substack{w(x)<p<x^\gamma\\ p\in\mathscr{P}_\flat}} \left(1-\frac{1}{p}\right),
\end{aligned}$$

and similarly for $\gamma\log x$ replaced by $\log Q$. These bounds yield

$$\begin{aligned}
&\Bigg| \sum_{Q_j,m_j,\delta_j,\delta'_j,d_j} \frac{J^*_j}{Q_j m_j d_j \operatorname{lcm}(\delta_j,\delta'_j)} \Bigg| \\
&\ll |(1+i\xi_5)(1+i\xi_6)| \prod_{\substack{p'>w(x)\\ p'\notin\mathscr{P}_\flat}} \left(1 + \frac{g(p')}{p'^{1+\frac{1}{\gamma\log x}}}\right) \\
&\quad \Bigg\{ \prod_{\substack{p>w(x)\\ p\in\mathscr{P}_\flat}} \left(1+\frac{h^\flat(p)}{p^{1+\frac{1}{\gamma\log x}}}\right)\left(1+\frac{1}{p^{1+\frac{2}{\gamma\log x}}}\right) \prod_{\substack{w(x)<p<x^\gamma\\ p\in\mathscr{P}_\flat}} \left(1-\frac{1}{p}\right)^{-2} \\
&\qquad + \sum_{Q\in\mathscr{P}_\flat} \frac{h^\flat(Q)}{Q^{1+\frac{1}{\gamma\log x}}} \prod_{\substack{w(x)<p<Q\\ p\in\mathscr{P}_\flat}} \left(1+\frac{h^\flat(p)}{p^{1+\frac{1}{\gamma\log x}}}\right)\left(1-\frac{1}{p}\right)^2 \prod_{\substack{w(x)<p''\\ p''\in\mathscr{P}_\flat}} \left(1+\frac{1}{p''^{1+\frac{2}{\log Q}}}\right) \Bigg\}.
\end{aligned}$$

Note that

$$\prod_{p\in\mathscr{A}} \left(1+\frac{a_p}{p^{1+\frac{c}{\log y}}}\right) \ll \prod_{\substack{p\in\mathscr{A}\\ p\leqslant y}} \left(1+\frac{a_p}{p}\right)$$

whenever $c \geqslant 1$ and the a_p, $p \in \mathscr{A}$, are non-negative and bounded. Thus, the above is bounded by

$$\begin{aligned}
&\ll |(1+i\xi_5)(1+i\xi_6)| \prod_{\substack{w(x)<p'<x^\gamma \\ p'\notin \mathscr{P}_\flat}} \left(1+\frac{g(p')}{p'}\right) \Bigg\{ \prod_{\substack{w(x)<p<x^\gamma \\ p\in \mathscr{P}_\flat}} \left(1+\frac{h^\flat(p)}{p}\right)\left(1-\frac{1}{p}\right) \\
&\qquad + \sum_{Q\in\mathscr{P}_\flat} \frac{h^\flat(Q)}{Q^{1+\frac{1}{\gamma\log x}}} \prod_{\substack{w(x)<p<Q \\ p\in\mathscr{P}_\flat}} \left(1+\frac{h^\flat(p)}{p^{1+\frac{1}{\gamma\log x}}}\right)\left(1-\frac{1}{p^{1+\frac{1}{\gamma\log x}}}\right)\Bigg\} \\
&\ll |(1+i\xi_5)(1+i\xi_6)| \prod_{\substack{w(x)<p'<x^\gamma \\ p'\notin \mathscr{P}_\flat}} \left(1+\frac{g(p')}{p'}\right) \prod_{\substack{w(x)<p<x^\gamma \\ p\in \mathscr{P}_\flat}} \left(1+\frac{h^\flat(p)}{p}\right)\left(1-\frac{1}{p}\right) \\
&\ll |(1+i\xi_5)(1+i\xi_6)| \prod_{w(x)<p<x^\gamma} \left(1+\frac{|h(p)|}{p}\right)\left(1-\frac{1}{p}\right) \\
&\ll_\gamma |1+i\xi_5||1+i\xi_6| \, \frac{\log w(x)}{\log x} \prod_{w(x)<p<x} \left(1+\frac{|h(p)|}{p}\right).
\end{aligned}$$

Integrating and taking the decay properties of the function θ and θ' into account yields the result. □

7. The average order of $\nu^\sharp\nu^\flat$

As a consequence of the proof of Proposition 6.1, we obtain the following lemma which shows that $\nu^\sharp\nu^\flat$ is of the correct average order on a positive proportion of the progressions $A \pmod{\widetilde{W}(T)}$ with $\gcd(A, W(T)) = 1$ provided (cf. the discussion at the end of Section 1) that $|S_h(T)| \asymp S_{|h|}(T)$.

Lemma 7.1. *We have*

$$|S_h(T;\widetilde{W}(T),A)| \ll S_{\nu^\sharp\nu^\flat}(T;\widetilde{W}(T),A) \ll \frac{\log w(T)}{\log T} \prod_{w(T)<p<T}\left(1+\frac{|h(p)|}{p}\right)$$

whenever $\gcd(A, W(T)) = 1$.

Proof. The first bound is immediate since $\nu^\sharp\nu^\flat$ is, outside of the sparse set $\mathscr{S}$, a majorant for $|h|$. The second bound follows from the upper bound (6.5) on

$$\frac{1}{T^s \operatorname{vol}\mathfrak{K}} \sum_{\mathbf{n}\in\mathbb{Z}^s\cap T\mathfrak{K}} \prod_{i=1}^{s} \nu_{h_i}(\phi_i(\mathbf{n})),$$

where we are only interested in the case $s = 1$. To see this, we apply first Lemma 6.2 and then Lemma 6.3 to the integrand, and finally recall that the outer sum (6.4) converges. □

8. The pretentious W-trick and orthogonality of h with nilsequences

In [15], the non-correlation of (W-tricked) multiplicative functions with nilsequences is proved for multiplicative functions which admit the W-trick recorded in Definition 8.1 below. In this section we establish this W-trick for all functions $f \in \mathscr{M}$.

It is an interesting question to investigate whether a suitably adapted version of this W-trick can be established for functions f that satisfy the condition (1.4) for $h = f$. Such a W-trick would need to take into account the rotation arising from the factor p^{it}.

Definition 8.1. Let $\kappa(x)$ and $\theta(x)$ be non-negative functions, which will be bounded in all applications. Let $\mathscr{F}_H(\kappa(x), \theta(x))$ denote the class of all multiplicative functions $f \in \mathscr{M}$ for which there exist functions $\varphi' : \mathbb{N} \to \mathbb{R}$ and $q^* : \mathbb{N} \to \mathbb{N}$ with the following properties:

(1) $\varphi'(x) \to 0$ as $x \to \infty$,
(2) $q^*(x)$ is $w(x)$-smooth and $q^*(x) \leqslant (\log x)^{\kappa(x)}$ for all $x \in \mathbb{N}$,
(3) if $x \in \mathbb{N}$ and if we set $\widetilde{W}(x) = q^*(x)W(x)$, then the estimate

$$\frac{q_0\widetilde{W}(x)}{|I|} \sum_{\substack{m \in I \\ m \equiv A \ (q_0\widetilde{W}(x))}} f(m) - S_f(x; \widetilde{W}(x), A) = O\left(\varphi'(x)\frac{1}{\log x}\frac{W(x)q}{\phi(W(x)q)} \prod_{\substack{p<x \\ p \nmid W(x)q}} \left(1 + \frac{|f(p)|}{p}\right)\right) \quad (8.1)$$

holds uniformly for all intervals $I \subseteq \{1, \ldots, x\}$ with $|I| > x(\log x)^{-\theta(x)}$, for all integers $0 < q_0 \leqslant (\log x)^{\theta(x)}$, and for all $A \in (\mathbb{Z}/q_0\widetilde{W}(x)\mathbb{Z})^*$.

The set $\mathscr{F}_H(\kappa(x), \theta(x))$ contains functions that have a W-trick where $\widetilde{W}$ is controlled by κ and where $\theta(x)$ is a measure for the quality of the major arc estimate.

8.1. **The elements of $\mathscr{M}$ admit a W-trick.**

We begin with the two lemmas that will reduce the task of proving (8.1) to that of bounding a restricted character sum. The first lemma is the following.

Lemma 8.2. *If $x > 1$, $f \in \mathscr{M}$ and $E > H \geqslant 1$, where H is as in Definition 1.1(i), then (8.1) follows for $\theta(x) = E$ if there is $\varphi'' = o(1)$ such that*

$$S_f(y; q_0\widetilde{W}(x), A) = S_f(y; \widetilde{W}(x), A) + O\left(\varphi''(x)\frac{Wq}{\phi(Wq)} \prod_{\substack{p<x \\ p \nmid Wq}} \left(1 + \frac{|f(p)|}{p}\right)\right) \quad (8.2)$$

for all $x > y \geqslant x(\log x)^{-E}$ and for all $A \in (\mathbb{Z}/q_0\widetilde{W}(x)\mathbb{Z})^*$. More precisely, we may take $\varphi'(x) = \varphi(x) + \varphi''(x) + (\log x)^{-E}$ in (8.1), where φ is as in Definition 1.1 (iv).

Proof. Suppose that $I = [y_1, y_1 + y_2] \subset [1, x]$ with $y_2 \geqslant x(\log x)^{-E}$ and note that condition (iv) of the Definition 1.1 implies that

$$S_f(y_1 + y_2; \widetilde{W}(x), A) = S_f(x; \widetilde{W}(x), A) + O\bigg(\varphi(x)\frac{Wq}{\phi(Wq)} \prod_{\substack{p<x \\ p\nmid Wq}} \Big(1 + \frac{|f(p)|}{p}\Big)\bigg).$$

Another application of Definition 1.1 (iv) shows that the left hand side of (8.1) satisfies

$$\begin{aligned} &\frac{q_0\widetilde{W}(x)}{|I|} \sum_{\substack{m\in I \\ m\equiv A \ (q_0\widetilde{W}(x))}} f(m) \qquad (8.3) \\ &= \frac{y_1+y_2}{y_2} S_f\Big(y_1 + y_2; q_0\widetilde{W}(x), A\Big) - \frac{y_1}{y_2} S_f\Big(y_1; q_0\widetilde{W}(x), A\Big) \\ &= \frac{y_1+y_2}{y_2} S_f\Big(x; q_0\widetilde{W}(x), A\Big) - \frac{y_1}{y_2} S_f\Big(y_1; q_0\widetilde{W}(x), A\Big) \\ &\qquad + O\bigg(\varphi(x)\frac{Wq}{\phi(Wq)} \prod_{\substack{p<x \\ p\nmid Wq}} \Big(1 + \frac{|f(p)|}{p}\Big)\bigg). \end{aligned}$$

We now split into two cases. If, on the one hand, $x > y_1 > y_2(\log x)^{-2E} > x(\log x)^{-3E}$, then Definition 1.1 (iv) shows that (8.3) is equal to

$$S_f\Big(x; q_0\widetilde{W}(x), A\Big) + O\left(\varphi(x)\frac{Wq}{\phi(Wq)} \prod_{\substack{p<x \\ p\nmid Wq}} \left(1 + \frac{|f(p)|}{p}\right)\right).$$

In this case, (8.1) follows from (8.2) and the above with $\varphi' = \varphi + \varphi''$.

If, on the other hand, $y_1 \leqslant y_2(\log x)^{-2E}$, then $\frac{y_1+y_2}{y_2} = (1 + O\left((\log x)^{-2E}\right))$ and it follows from $E > H$ that

$$\begin{aligned} \frac{y_1}{y_2} S_f\Big(y_1; q_0\widetilde{W}(x), A\Big) &\leqslant \frac{y_1}{y_2}(\log x)^{H-1} \leqslant (\log x)^{-E-1} \\ &\leqslant (\log x)^{-E} \frac{Wq}{\phi(Wq)} \prod_{\substack{p<x \\ p\nmid Wq}} \Big(1 + \frac{|f(p)|}{p}\Big). \end{aligned}$$

Thus, (8.3) equals in this case

$$S_f(x; q_0\widetilde{W}(x), A) + O\bigg(\Big(\varphi(x) + (\log x)^{-E}\Big) \frac{Wq}{\phi(Wq)} \prod_{\substack{p<x \\ p\nmid Wq}} \Big(1 + \frac{|f(p)|}{p}\Big)\bigg)$$

and an application of (8.2) yields (8.1) with $\varphi'(x) = \varphi(x) + \varphi''(x) + (\log x)^{-E}$. □

Following the above reduction, we now proceed to analyse the difference of the two mean values that appear in (8.2).

Lemma 8.3 (Restricted character sum)**.** *Suppose that* $x \geqslant y \geqslant x(\log x)^{-E}$*, let* $q_0 \geqslant 1$ *be an integer and suppose* $\gcd(A, q_0\widetilde{W}(x)) = 1$*. Then*

$$S_f(y;\widetilde{W},A) - S_f(y;q_0\widetilde{W},A) = \frac{q_0\widetilde{W}}{x}\frac{1}{\phi(q_0\widetilde{W})}\sideset{}{^*}\sum_{\chi \pmod{q_0\widetilde{W}}} \chi(A) \sum_{n\leqslant x} f(n)\overline{\chi}(n), \tag{8.4}$$

where $\sum^*$ *indicates the restriction of the sum to characters that are not induced from characters* $\pmod{\widetilde{W}}$*.*

Proof. We have

$$\begin{aligned}
&S_f(y;\widetilde{W},A) - S_f(y;q_0\widetilde{W},A)\\
&= \frac{\widetilde{W}}{y}\Bigg(\sum_{\substack{n\leqslant y\\ n\equiv A \pmod{\widetilde{W}}}} f(n) - q_0 \sum_{\substack{n\leqslant y\\ n\equiv A \pmod{q_0\widetilde{W}}}} f(n)\Bigg)\\
&= \frac{1}{y}\frac{\widetilde{W}}{\phi(q_0\widetilde{W})} \sum_{\chi \pmod{q_0\widetilde{W}}} \Bigg(\sum_{\substack{A' \pmod{q_0\widetilde{W}}\\ A\equiv A' \pmod{\widetilde{W}}}} \chi(A') - q_0\chi(A)\Bigg) \sum_{n\leqslant y} f(n)\overline{\chi}(n)\\
&= \frac{1}{y}\frac{\widetilde{W}}{\phi(q_0\widetilde{W})} \sideset{}{^*}\sum_{\chi \pmod{q_0\widetilde{W}}} \Bigg(\sum_{\substack{A' \pmod{q_0\widetilde{W}}\\ A\equiv A' \pmod{\widetilde{W}}}} \chi(A') - q_0\chi(A)\Bigg) \sum_{n\leqslant y} f(n)\overline{\chi}(n),
\end{aligned} \tag{8.5}$$

where $\sum^*$ indicates the restriction of the sum to characters that are not induced from characters $\pmod{\widetilde{W}}$; for all other characters we have $\chi(A') = \chi(A)$ and the difference in the brackets above is zero. It remains to show that the sum over A' in (8.5) vanishes. However,

$$\sum_{\substack{A' \pmod{q_0\widetilde{W}}\\ A\equiv A' \pmod{\widetilde{W}}}} \chi(A') = \frac{1}{\phi(\widetilde{W})} \sum_{\chi' \pmod{\widetilde{W}}} \overline{\chi'}(A) \sum_{A' \pmod{q_0\widetilde{W}}} \chi(A')\chi'(A') = 0,$$

since $\chi\chi'$ is a non-trivial character modulo $q_0\widetilde{W}$. Thus the lemma follows. □

We aim to exploit the fact that the character sum on the right hand side of (8.4) is restricted by means of the following consequence of the 'pretentious large sieve':

Proposition 8.4 (Granville and Soundararajan [9])**.** *Let* $C > 0$ *be fixed and let* f *be a bounded multiplicative function. For any given* x*, consider the set of primitive characters of conductor at most* $(\log x)^C$ *and enumerate them as* $\chi_1, \chi_2, \ldots$ *in such a way that* $|S_{f\overline{\chi}_1}(x)| \geqslant |S_{f\overline{\chi}_2}(x)| \geqslant \ldots$ *. If* x *is sufficiently large, then the following holds for all* $x^{1/2} \leqslant X \leqslant x$

and $q \leqslant (\log x)^C$. *Let* $\mathscr{C}$ *be any set of characters modulo* q, $q \leqslant (\log x)^C$, *which does not contain characters induced by* $\chi_1, \dots, \chi_k$, *where* $k \geqslant 2$. *Then*

$$\left|\frac{1}{\phi(q)} \sum_{\chi \in \mathscr{C}} \chi(a) \sum_{n \leqslant X} f(n)\overline{\chi}(n)\right|$$
$$\ll_C \frac{e^{O_C(\sqrt{k})}X}{q} \left(\frac{\log\log x}{\log x}\right)^{1-\frac{1}{\sqrt{k}}} \log\left(\frac{\log x}{\log\log x}\right) \prod_{p \leqslant q, p \nmid q} \left(1 + \frac{|f(p)|-1}{p}\right).$$

Equipped with these tools, we are now ready to state and prove the two main results of this section.

Proposition 8.5. *Suppose that* $f \in \mathscr{M}$ *and let* H *be the parameter from Definition 1.1 (i). Let* $d, m_G \geqslant 1$ *be integers and let* G/Γ *be a nilmanifold of dimension* m_G*, equipped with a filtration of length d. Then there exist, for any positive* x*, positive integers* $\kappa(x)$ *and* $E(x)$ *such that* $\kappa(x), E(x) \ll_{d,m_G,H} 1$*, such that* $E(x)$ *is sufficiently large with respect to* d, m_G, H *and* $\kappa(x)$ *for* [15, Theorem 5.1] *to apply to* G/Γ*, and such that* $f \in \mathscr{F}_H(\kappa(x), E(x))$*. More precisely,* (8.1) *holds with* $\varphi'(x) = \varphi(x) + (\log x)^{-\alpha_f/(3H)} + (\log x)^{-E(x)}$*, where* φ *is as in (iv) of Definition 1.1.*

The following corollary summarises an immediate consequence of the above proposition and [15, Theorem 5.1]. This is the result that will be used in the proof of Proposition 2.1 in the final section.

Corollary 8.6. *Let* f *and* G/Γ *be as in Proposition 8.5 and let* $\widetilde{W}(x)$ *be as guaranteed by the above conclusion that* $f \in \mathscr{F}_H(\kappa(x), E(x))$*. Suppose that* G *is* $r-2$*-step nilpotent, let* $G_\bullet$ *be the filtration of length* d *and suppose that* G/Γ *has a* Q_0*-rational Mal'cev basis adapted to this filtration. Then the following estimate holds for all* $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ *and for all* 1*-bounded Lipschitz functions* $F : G/\Gamma \to \mathbb{C}$

$$\left|\frac{\widetilde{W}}{T} \sum_{n \leqslant T/\widetilde{W}} (f(\widetilde{W}n + A) - S_f(T; \widetilde{W}, A))F(g(n)\Gamma)\right| \tag{8.6}$$
$$\ll_{\|F\|_{\mathrm{Lip}}} \left\{\varphi(N) + \frac{1}{\log w(N)} + \frac{Q_0^{O_{d,m_G}(1)}}{(\log\log T)^{1/(2^{r+2}\dim G)}}\right\} \frac{1}{\log T} \frac{W}{\phi(W)} \prod_{\substack{p<T \\ p \nmid W(N)}} \left(1 + \frac{|f(p)|}{p}\right),$$

where φ *is as in Definition 1.1 (iv). The implied constants depend, in addition to* $\|F\|_{\mathrm{Lip}}$*, also on* H*, on the step and dimension of* G*, on* Q_0 *and on the degree of the filtration* $G_\bullet$*.*

We now turn to the proof of the main result of this section.

Proof of Proposition 8.5. To start with, assume that $|f(p)| \leqslant 1$ for all primes p and let α_f be as in Definition 1.1 (iii). Setting $\varepsilon := \frac{1}{2}\min(1, \alpha_f/2)$ and $k := \lceil \varepsilon^{-2} \rceil \geqslant 2$, we aim to apply Proposition 8.4 with this value of k to bound (8.4). That is, our task is to show that there are suitable choices of $\widetilde{W}(x)$ and E so that E is large enough for [15, Theorem 5.1] to apply and so that Proposition 8.4 applies to all sets of characters that appear in (8.4).

For this purpose, we define two sequences of integers, $H_0, H_1, \dots, H_k$ and $E_0, E_1, \dots, E_k$, as follows. Let $H_0 = 1$. If H_i is defined, let $E_i \ll_{d,m_G,H_i} 1$ be such that E_i is sufficiently large for [15, Theorem 5.1] to apply for any $r-2$-step nilmanifold. If H_i and E_i are defined, set $H_{i+1} = 2H_i + E_i + 1$.

Let $C = 2H_k + E_k + 2$ and let $\chi_1, \dots, \chi_k$ be the characters of conductor at most $(\log x)^C$ that are defined in the statement of Proposition 8.4. We now define $k+1$ disjoint intervals $[2^{r_i}W(x), 2^{r_i}W(x)(\log x)^{E_i}]$, $0 \leqslant i \leqslant k$, where $r_i = \lceil \log_2((\log x)^{H_i}) \rceil$. Since $W(x) < (\log x)^2$, these are all contained in $[1, (\log x)^C]$. Thus, there exists an index $j \in \{0, \dots, k\}$ such that the interval $[2^{r_j}W(x), 2^{r_j}W(x)(\log x)^{E_j}]$ does not contain the conductor of any of the characters $\chi_1, \dots, \chi_k$.

Setting $q^*(x) = 2^{r_j}$ and $E = E_j$, we note that the conductor of each character in the sum (8.4) belongs to $[q^*(x)W(x), q^*(x)W(x)(\log x)^E]$. Hence Proposition 8.4 shows that

$$\begin{aligned} S_f(x; \widetilde{W}, A) - S_f(x; q_0\widetilde{W}, A) &= \frac{1}{x}\frac{q_0\widetilde{W}}{\phi(q_0\widetilde{W})} \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q_0\widetilde{W})} \chi(A) \sum_{n \leqslant x} f(n)\overline{\chi}(n) \\ &\ll_{d,m_G,H,\alpha_f} \left(\frac{\log\log x}{\log x}\right)^{1-\frac{1}{\sqrt{k}}} \log\left(\frac{\log x}{\log\log x}\right) \\ &\ll_{d,m_G,H,\alpha_f} (\log x)^{-1+\alpha_f/2}(\log\log x)^2 \\ &\ll_{d,m_G,H,\alpha_f} \frac{(\log\log x)^2}{(\log x)^{\alpha_f/2}} \frac{1}{\log x} \frac{Wq}{\phi(Wq)} \exp\left(\sum_{\substack{p \leqslant x \\ p \nmid Wq}} \frac{|f(p)|}{p}\right). \end{aligned}$$

To handle the general case, we decompose f as $f = g^{*H} * g'$, with g as in (1.1) for $h = f$. Let $\varepsilon_H = \min(1, \alpha_f/H)/2$ so that the above applies to g with ε replaced by ε_H and with α_f replaced by α_f/H. Keeping in mind that g' is supported in square-full numbers only, we obtain the following estimate by combining the above with an application of the hyperbola method:

$$\begin{aligned} &\frac{q_0\widetilde{W}}{x}\frac{1}{\phi(q_0\widetilde{W})} \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q_0\widetilde{W})} \chi(A) \sum_{n \leqslant x} f(n)\overline{\chi}(n) \\ &\ll_H \sum_{D \leqslant x^{1-1/H}} \sum_{d_1 \dots d_{H-1} = D} \frac{g(d_1)\dots g(d_{H-1})|\overline{\chi}(D)|}{D} \sum_{i=1}^{H} \\ &\qquad \frac{1}{x}\frac{q_0\widetilde{W}D}{\phi(q_0\widetilde{W})} \left| \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q_0\widetilde{W})} \chi(A) \sum_{\substack{n: \\ x^{1-1/H} \max(d_1,\dots,d_{i-1}) \\ \leqslant Dn \leqslant x}} g(n)\overline{\chi}(n) \right| \\ &\ll_{d,m_G,H,\alpha_f} \frac{(\log\log x)^2}{(\log x)^{\alpha_f/(2H)}} \frac{1}{\log x} \frac{Wq}{\phi(Wq)} \exp\left(\sum_{\substack{p \leqslant x \\ p \nmid Wq}} \frac{|f(p)|}{p}\right). \end{aligned}$$

This yields (8.2) with $\varphi''(x) = (\log x)^{-\alpha_f/(3H)}$. Hence, Lemma 8.2 implies the result. □

9. Conclusion of the proof of Proposition 2.1

We now have everything in place to complete the proof of Proposition 2.1 and, thus, Theorem 1.2. We begin by noting that Proposition 2.1 holds trivially unless

$$|S_{h_i}(T, \widetilde{W}(T), A_i)| \asymp \frac{\log w(T)}{\log T} \prod_{w(T)<p\leqslant T} \left(1 + \frac{|h_i(p)|}{p}\right) \tag{9.1}$$

for all $1 \leqslant i \leqslant r$. Assuming (9.1) from now on, it follows from Corollary 8.6 and the inverse theorem for uniformity norms from [11] that the function

$$\tilde{h}_i : n \mapsto \frac{h_i(\widetilde{W}(T)n + A_i) - S_{h_i}(T, \widetilde{W}(T), A_i)}{S_{|h_i|}(T, \widetilde{W}(T), A_i)},$$

defined for positive integers $n \leqslant T/\widetilde{W}(T)$, satisfies

$$\|\tilde{h}_i\|_{U^{r-1}} = o(1).$$

By Lemma 7.1, the majorant $\nu_i^\flat \nu_i^\sharp$ for h_i is of the correct order. Thus,

$$\tilde{\nu}_i = \frac{1}{2}\left(\frac{\nu_i^\flat \nu_i^\sharp}{S_{h_i}(T, \widetilde{W}(T), A_i)} + \mathbf{1}\right)$$

is a correct-order majorant for $\tilde{h}_i$. Furthermore, Proposition 6.1 shows that this majorant is pseudo-random. Proposition 2.1 now follows from the generalised von Neumann theorem [10, Proposition 7.1].

10. Proof of Theorem 1.3 and Corollary 1.4

To deduce Theorem 1.3 from Theorem 1.2, we first observe that Proposition 8.4 and Lemma 1.5 imply that

$$\begin{aligned} S_{h_i}(T, \widetilde{W}(T), A_i) &= \frac{\widetilde{W}}{\phi(\widetilde{W})} \sum_{\chi_i \in \mathscr{E}_i} \overline{\chi_i}(A_i) \frac{1}{T} \sum_{n \leqslant T} h_i(n)\chi_i(n) + o\left(S_{|h_i'|}(T) \prod_{p \leqslant w(T)} \left(1 - \frac{1}{p}\right)^{-1}\right) \\ &= \sum_{\chi_i \in \mathscr{E}_i} S_{|h_i'|}(T) \prod_{p \leqslant w(T)} \frac{\overline{\chi_{i,p}}(A_i)}{1 - p^{-1}} \prod_{w(T) \leqslant p' \leqslant T} \left(\frac{1 + \chi_i(p')h_i(p')p'^{-1} + \dots}{1 + |h_i(p')|p'^{-1} + \dots}\right) \\ &\quad + o\left(S_{|h_i'|}(T) \prod_{p \leqslant w(T)} \left(1 - \frac{1}{p}\right)^{-1}\right), \end{aligned}$$

where $h_i'(n) = h_i(n)\mathbf{1}_{\gcd(n,W(T))=1}$. When combined with a truncation argument as in §2, the stability condition (iv) from Definition 1.1 yields furthermore

$$S_{|h_i'|}(T) = (1 + o(1))S_{|h_i|}(T) \prod_{p \leqslant w(T)} \left(1 + \frac{|h_i(p)|}{p} + \dots\right)^{-1},$$

which can be inserted into the previous expression.

If $\varphi(\mathbf{v}) \equiv A_i w_i \pmod{p^{v_p(w_i\widetilde{W})}}$ for all $p \leqslant w(T)$, then

$$\overline{\chi_i}(A_i) = \overline{\chi_i}\left(\frac{\varphi_i(\mathbf{v})}{w_i}\right) = \prod_{p\leqslant w(T)} \overline{\chi_{i,p}}\left(\frac{\varphi_i(\mathbf{v})}{w_i}\right) = \prod_{p\leqslant w(T)} \overline{\chi_{i,p}}\left(\frac{\varphi_i(\mathbf{v})}{p^{v_p(w_i)}}\right) \prod_{q\neq p} \chi_{i,p}(p^{v_p(w_i)}).$$

Thus, with the help of the Chinese Remainder Theorem, we obtain

$$\sum_{\substack{w_1,\dots,w_r \\ p|w_i \Rightarrow p<w(T) \\ w_i \leqslant (\log T)^{B_2}}} \sum_{\substack{A_1,\dots,A_r \\ \in(\mathbb{Z}/\widetilde{W}\mathbb{Z})^*}} \left(\prod_{i=1}^r h_i(w_i)\overline{\chi_i}(A_i)\right) \frac{1}{(w\widetilde{W})^s} \sum_{\substack{\mathbf{v}\in \\ (\mathbb{Z}/w\widetilde{W}\mathbb{Z})^s}} \prod_{j=1}^r \mathbf{1}_{\varphi_j(\mathbf{v})\equiv w_j A_j \pmod{w_j\widetilde{W}}}$$

$$= (1+o(1)) \prod_{p<w(T)} \lim_{m\to\infty} \frac{1}{p^{ms}} \sum_{\mathbf{a}\in\mathbb{N}_0} \sum_{\substack{\mathbf{v}\in(\mathbb{Z}/p^m\mathbb{Z})^s \\ v_p(\varphi_i(\mathbf{v}))=a_i}} \prod_{j=1}^r h_j(p^{a_j})\overline{\chi_{i,p}}\left(\frac{\varphi_i(\mathbf{v})}{p^{v_p(w_i)}}\right) \prod_{q\neq p} \chi_{i,p}(p^{v_p(w_i)}).$$

Finally, we observe that for $p \geqslant w(T)$ with T sufficiently large:

$$\lim_{m\to\infty} \frac{1}{p^{ms}} \sum_{\mathbf{a}\in\mathbb{N}_0} \sum_{\substack{\mathbf{v}\in(\mathbb{Z}/p^m\mathbb{Z})^s \\ v_p(\varphi_i(\mathbf{v}))=a_i}} \prod_{j=1}^r h_j(p^{a_j})\overline{\chi_i}(p^{a_j})$$
$$= \prod_{j=1}^r \left(1-\frac{1}{p}\right)\left(1+\frac{h_j(p)\overline{\chi_i}(p)}{p}\right) + O_r(p^{-2}H^2), \qquad (10.1)$$

which follows from [3, (5.6)] with $M = 1$ by arguing as in [3, §5.2]. Theorem 1.3 now follows by inserting all the above information into the expression (1.2).

Corollary 1.4 follows fairly straightforwardly from (1.2) by inserting the assumptions of the corollary and by applying an identity of the form (10.1) but with the factors $\overline{\chi_j}(p^{a_i})$ and $\overline{\chi_j}(p)$ all removed, i.e. replaced by 1.

11. Application to eigenvalues of cusp forms

In this section we show that Theorem 1.2 applies to the normalised eigenvalues of holomorphic cusp forms. Let f be a primitive cusp form of weight $k \geqslant 2$ and level $N \geqslant 1$. Then f has a Fourier expansion of the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz),$$

where λ_f is multiplicative and satisfies Deligne's bound

$$|\lambda_f(n)| \leqslant d(n),$$

where d is the divisor function.

Lemma 11.1. $|\lambda_f(n)| \in \mathscr{M}$. *Thus,* $h_i = |\lambda_{f_i}(n)|$ *is a permissible choice in Theorem 1.2.*

Proof. Conditions (i) and (ii) of Definition 1.1 follow directly from the properties summarised above. Condition (iii) follows from Rankin [16, Theorem 2] since

$$\sum_{p\leqslant x}|\lambda_f(p)|\log p \geqslant \frac{1}{2}\sum_{p\leqslant x}\lambda_f(p)^2\log p \sim \frac{x}{2}.$$

for large x. Thus, it remains to show that condition (iv) holds too. For this purpose, we will first deduce from the Sato–Tate law that (iii) in fact holds for some $\alpha_{|\lambda_f|} > 2/\pi$. This will allow us later to employ the Lipschitz bounds [9] for multiplicative functions. Note that the Sato–Tate conjecture [2] implies:

$$\#\{p \leqslant x : 0 \leqslant |\lambda_p| \leqslant \alpha\}\log x \sim x\mu(\alpha),$$

where

$$\mu(\alpha) = \frac{2}{\pi}\arcsin(\alpha/2) + \frac{1}{\pi}\sin(2\arcsin(\alpha/2)).$$

for $\alpha \in [0, 2]$. Thus,

$$\begin{aligned}\sum_{p\leqslant x}|\lambda_f(p)|\log p &\geqslant \sum_{x^{1-\varepsilon}<p\leqslant x}(1-\varepsilon)|\lambda_f(p)|\log x\\ &\geqslant (1-\varepsilon)\sum_{n=1}^{N}\sum_{\substack{x^{1-\varepsilon}<p\leqslant x\\ 2(n-1)/N\leqslant|\lambda_f(p)|\leqslant 2n/N}}\frac{n-1}{N}\log x\\ &\sim x(1-\varepsilon)\sum_{n=1}^{N}\frac{2(n-1)}{N}\Big(\mu(2(n-1)/N)-\mu(2n/N)\Big).\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small and N sufficiently large, the claimed lower bound on $\alpha_{|\lambda_f|}$ now follows, since

$$\int_0^2 \alpha\,\mathrm{d}\mu(\alpha) = 2\mu(2) - \int_0^2\left(\frac{2}{\pi}\arcsin(\alpha/2) + \frac{1}{\pi}\sin(2\arcsin(\alpha/2))\right)\mathrm{d}\alpha = \frac{8}{3\pi} > \frac{2}{\pi}.$$

If g is the multiplicative functions defined by

$$g(p^k) = \begin{cases}|\lambda_f(p)|/2 & \text{if } k=1\\ 0 & \text{if } k>1\end{cases},$$

then (iii) holds for g with some $\alpha_g > \frac{1}{\pi}$, and, hence, $\sum_{n\leqslant x} g(n) \gg x(\log x)^{-1+1/\pi+c}$ for some $c > 0$. Thus, if $x' \in (x(\log x)^{-C}, x]$, then the Lipschitz bounds [8] of Granville and Soundararajan imply that

$$\begin{aligned}\frac{1}{x}\sum_{n\leqslant x}g(n) &= \frac{1}{x'}\sum_{n\leqslant x'}g(n) + O_{C,\varepsilon'}\left((\log x)^{-1+\frac{1}{\pi}+\varepsilon'}\right) \qquad (11.1)\\ &= \frac{1}{x'}\sum_{n\leqslant x'}g(n) + o\left(\frac{1}{x}\sum_{n\leqslant x}g(n)\right).\end{aligned}$$

If $\chi \pmod q$ is a character for which there exists $t = t_\chi \in \mathbb{R}$ such that (1.4) holds for $h(n) = g(n)\chi(n)$, then Elliott's asymptotic formula (1.5), combined with (11.1), shows that

$$\begin{aligned} &\frac{1}{x}\sum_{n\leqslant x} g(n)(\overline{\chi}(A)\chi(n) + \chi(A)\overline{\chi}(n)) \\ &= S_g(x)\cdot 2\Re\left(\frac{\chi(A)}{1-it}x^{-it}\prod_{p\leqslant x}\left(\frac{1+g(p)\chi(p)p^{it-1}+\dots}{1+g(p)p^{-1}+\dots}\right)\right) + o(S_g(x)) \\ &= S_g(x')\cdot 2\Re\left(\frac{\chi(A)}{1-it}x^{-it}\prod_{p\leqslant x}\left(\frac{1+g(p)\chi(p)p^{it-1}+\dots}{1+g(p)p^{-1}+\dots}\right)\right) + o(S_g(x)) \\ &= \frac{1}{x'}\sum_{n\leqslant x'} g(n)(\overline{\chi}(A)\chi(n) + \chi(A)\overline{\chi}(n)) + o(S_g(x)). \end{aligned} \tag{11.2}$$

If no such t exists for χ, then Lemma 1.5 yields

$$\frac{1}{x'}\sum_{n\leqslant x'} g(n)(\overline{\chi}(A)\chi(n) + \chi(A)\overline{\chi}(n)) = o(S_g(x)). \tag{11.3}$$

for all x' as above. In order to employ these facts, as well as Proposition 8.4, which is also a statement about bounded multiplicative functions, we shall now decompose $S_{|\lambda_f|}(x;q,A)$ into character sums involving the bounded function g. Since $|\lambda_f(n)| = g*g*g'$, where $g'(n)$ is zero unless n is square-full and satisfies $|g'(p^k)| \leqslant C'^k$ for some constant C', we have

$$\begin{aligned} &S_{|\lambda_f|}(x;q,A) \\ &= \frac{1}{\phi(q)}\sum_{\chi \pmod q}\chi(A)\frac{1}{x}\sum_{n\leqslant x}|\lambda_f(n)|\overline{\chi}(n) \\ &= \sum_{\substack{n_0\leqslant x^{1/2}\\(n_0,q)=1}}\sum_{\substack{n_1\leqslant (x/n_0)^{1/2}\\(n_1,q)=1}} g'(n_0)g(n_1)\frac{1}{\phi(q)}\sum_{\chi \pmod q}\chi(A\overline{n_0n_1})\chi(A)\frac{1}{x}\sum_{\substack{n_2\leqslant x/(n_0n_1)\\(n_2,q)=1}} g(n_2)\overline{\chi}(n_2) \\ &\quad + \sum_{\substack{n_0\leqslant x^{1/2}\\(n_0,q)=1}}\sum_{\substack{n_1< (x/n_0)^{1/2}\\(n_1,q)=1}} g'(n_0)g(n_1)\frac{1}{\phi(q)}\sum_{\chi \pmod q}\chi(A\overline{n_0n_1})\frac{1}{x}\sum_{\substack{(x/n_0)^{1/2}<n_2\leqslant x/(n_0n_1)\\(n_2,q)=1}} g(n_2)\overline{\chi}(n_2) \\ &\quad + O(x^{-1/12}), \end{aligned} \tag{11.4}$$

where $\overline{n}$ denotes the inverse modulo q, and where the truncation of the summation in n_0 is justified as follows. Observe that the bound $|\lambda_f(n)| \leqslant d(n) \ll_\varepsilon n^\varepsilon$ implies that $g'(n) \ll_\varepsilon n^\varepsilon$. This follows inductively for $n = p^k$, since $|\lambda_f(p^k)| = g'(p^k) + 2g'(p^{k-1})g(p) + g'(p^{k-2})g(p)^2$. Observing further that for any square-full integer $n_0 > y$ its largest square divisor q^2

satisfies $q > y^{1/3}$, we have

$$\sum_{\substack{x^{1/2}<n_0\leqslant x\\(n_0,q)=1}} \frac{g'(n_0)}{n_0} S_{g*g}\left(\frac{x}{n_0}; q, A\overline{n_0}\right) \ll \sum_{\substack{x^{1/2}<n_0\leqslant x\\(n_0,q)=1}} \frac{g'(n_0)}{n_0} \ll_\varepsilon \sum_{x^{1/2}<n_0\leqslant x} \frac{1}{n_0^{1-\varepsilon}}$$

$$\ll \sum_{x^{1/6}<q\leqslant x^{1/2}} \frac{1}{q^{2-2\varepsilon}} \prod_{p\mid q}\left(1+\frac{1}{p^{1-\varepsilon}}\right) \ll \sum_{q>x^{1/6}} \frac{d(q)}{q^{2-2\varepsilon}} \ll_\varepsilon \sum_{q>x^{1/6}} \frac{1}{q^{2-3\varepsilon}} \ll \frac{1}{x^{(1-3\varepsilon)/6}},$$

which is seen to be $O(x^{-1/12})$ if ε was chosen sufficiently small.

Proposition 8.4, applied with $k = \lceil \pi^{1/2} \rceil$ and $q \leqslant (\log x)^C$, furthermore shows that, if $\mathscr{E}_q \subset \{\chi_1, \dots, \chi_k, \overline{\chi}_1, \dots, \overline{\chi}_k\}$ is the subset of all characters with conductor q, then (11.4) equals

$$\begin{aligned}
&S_{|\lambda_f|}(x; q, A)\\
&= \sum_{\substack{n_0\leqslant x^{1/2}\\(n_0,q)=1}} \sum_{\substack{n_1\leqslant (x/n_0)^{1/2}\\(n_1,q)=1}} g'(n_0)g(n_1)\frac{1}{\phi(q)} \sum_{\chi\in\mathscr{E}_q} \chi(A\overline{n_0n_1})\chi(A)\frac{1}{x} \sum_{\substack{n_2\leqslant z/(n_0n_1)\\(n_2,q)=1}} g(n_2)\overline{\chi}(n_2)\\
&\quad + \sum_{\substack{n_0\leqslant x^{1/2}\\(n_0,q)=1}} \sum_{\substack{n_1< (x/n_0)^{1/2}\\(n_1,q)=1}} g'(n_0)g(n_1)\frac{1}{\phi(q)} \sum_{\chi\in\mathscr{E}_q} \chi(A\overline{n_0n_1})\frac{1}{x} \sum_{\substack{(x/n_0)^{1/2}<n_2\leqslant z/(n_0n_1)\\(n_2,q)=1}} g(n_2)\overline{\chi}(n_2)\\
&\quad + o\left(\frac{q}{\phi(q)}\frac{1}{\log x}\prod_{\substack{p\leqslant x\\p\nmid q}}\left(1+\frac{|\lambda_f(p)|}{p}\right)\right),
\end{aligned}$$

where $z = x$ in the sums over n_2. Note that $\#\mathscr{E}_q \leqslant 2k \ll 1$ and that we always have $z/(n_0n_1) \geqslant zx^{-3/4}$ in the expression above. Thus, by splitting the second sum over n_2 into a difference of two sums, and applying afterwards (11.2) or (11.3), repectively, to those two sums that run up to $z/(n_0n_1)$, we deduce that the above continues to hold for every $z \in (x(\log x)^{-C}, x)$, with an error term of the exact same shape. In view of (11.4), the above equals, however,

$$S_{|\lambda_f|}(z; q, A) + o\left(\frac{q}{\phi(q)}\frac{1}{\log x}\prod_{\substack{p\leqslant x\\p\nmid q}}\left(1+\frac{|\lambda_f(p)|}{p}\right)\right),$$

which is the content of property (iv) of Definition 1.1 and shows that $|\lambda_f| \in \mathscr{M}$. □

12. Further applications

In this section we sketch the proofs of two applications that follow from the Selberg–Delange method.

Lemma 12.1. *Let $\delta \in (0, 1)$ and let $h(n) = \delta^{\omega(n)}$, where $\omega(n)$ denotes the number of distinct prime factors of n. Then $h \in \mathscr{M}$.*

Proof. Conditions (i)–(iii) are immediate. Thus, let x, x' and q be as in Definition 1.1 (iv). The Selberg–Delange method allows us to deduce an asymptotic formula for

$$\sum_{n\leqslant x} h(n)\chi(n)$$

for any character $\chi \pmod q$, by relating the Dirichlet series $\sum_{n\geqslant 1} h(n)\chi(n)n^{-s}$ to $L(s,\chi)^{\delta}$. The shape of this asymptotic formula implies that either both

$$\frac{1}{x}\sum_{n\leqslant x} h(n)\chi(n) = O(e^{-c\sqrt{\log x}}) \quad \text{and} \quad \frac{1}{x'}\sum_{n\leqslant x'} h(n)\chi(n) = O(e^{-c\sqrt{\log x}})$$

are small, or

$$\frac{1}{x}\sum_{n\leqslant x} h(n)\chi(n) \sim \frac{1}{x'}\sum_{n\leqslant x'} h(n)\chi(n).$$

Hence, $S_h(x;q,a) \sim S_h(x';q,a)$, as required. □

Lemma 12.2. *Let $K/\mathbb{Q}$ be a finite Galois extension and let $h(n)$ denote the characteristic function of integers that are composed of primes p which split completely over K. Then $h \in \mathscr{M}$.*

Proof. Conditions (i) and (ii) are immediate. Condition (iii) follows from the Chebotarev density theorem. To verify (iv), suppose that $[K:\mathbb{Q}] = d$, and let $\mathscr{P}$ denote the set of rational primes that split completely in K. Let

$$F(s) = \prod_{p\in\mathscr{P}} \left(1 - dp^{-s}\right)^{-1}$$

for $\Re s > 1$. Then there is a function $G(s)$ that is holomorphic and non-zero in $\Re s > \frac{1}{2}$ and such that $\zeta_K(s) = F(s)G(s)$, thus $F(s)$ has a meromorphic continuation to $\Re s > \frac{1}{2}$. Condition (iv) then follows as above from the Selberg–Delange method. In this case, we relate $\prod_{p\in\mathscr{P}} \left(1-p^{-s}\right)^{-1}$ to $F(s)^{1/d}$, that is, one needs to analyse

$$\prod_{p\in\mathscr{P}} \left(1-p^{-s}\right)^{-1}\left(1-dp^{-s}\right)^{1/d} = \prod_{p\in\mathscr{P}} \left(1-p^{-s}\right)^{-1} G(s)^{1/d}\zeta_K(s)^{-1/d},$$

or, if $\chi \pmod q$ is a Dirichlet character, then we consider instead

$$\prod_{p\in\mathscr{P}} \left(1-\chi(p)p^{-s}\right)^{-1}\left(1-d\chi(p)p^{-s}\right)^{1/d} = \prod_{p\in\mathscr{P}} \left(1-\chi(p)p^{-s}\right)^{-1} G_\chi(s)^{1/d}L(\xi,s)^{-1/d},$$

where G_χ is the twist of G by χ and where ξ is a suitable Hecke character. □

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