Note
There are uncountably many topological types of locally finite trees

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Abstract

Consider two locally finite rooted trees as equivalent if each of them is a topological minor of the other, with an embedding preserving the tree-order. Answering a question of van der Holst, we prove that there are uncountably many equivalence classes.

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Let the *tree-order* $\leq$ on the set of vertices of a rooted tree T be defined by setting $x \leq y$ for vertices x, y iff x lies on the unique path in T from its root to y. Let us call two locally finite rooted trees *equivalent* if each of them is a topological minor of the other, with an embedding that respects the tree-order. Call the equivalence classes *topological types* of such trees. The purpose of this note is to answer a question raised by van der Holst [2], by proving that there are uncountably many topological types of locally finite trees.

Our proof uses Nash-Williams's theorem that the – finite or infinite – rooted trees are well-quasi-ordered under this relation: See Nash-Williams [4], or Kühn [3] for a short proof. An introduction to the well-quasi-ordering of trees can be found in Diestel [1, Ch.12]. For all terms that remain undefined here we refer to [1].

Theorem 1 *There are uncountably many topological types of locally finite trees.*

In the remainder of this note we prove Theorem 1. By $\preccurlyeq$ we denote the topological minor relation between rooted trees that respects their tree-order. Let $\mathcal{T}$ be a class of rooted trees. A tree T is said to be *universal* with respect to $(\mathcal{T}, \preccurlyeq)$, if $T \in \mathcal{T}$ and $X \preccurlyeq T$ for every $X \in \mathcal{T}$.

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Note that the number of topological types of locally finite trees is not finite, since for instance any two finite trees of different order are of different topological type. Suppose there are only countably many topological types of locally finite trees. Let $\mathcal{G}_1$ be a set of locally finite trees, exactly one of each isomorphism type. Our aim is to construct a bad sequence $T_1, T_2, \dots$ of trees in $\mathcal{G}_1$, i.e., a sequence $T_1, T_2, \dots$ such that $T_i \not\preccurlyeq T_j$ whenever $i < j$.

To start with, choose trees $X_1^1, X_2^1, \dots$ from $\mathcal{G}_1$, exactly two trees of each topological type. Consider the disjoint countable union $X_1^1 \cup X_2^1 \cup \dots$ and let $R^1 = w_1^1 w_2^1 \dots$ be an additional ray. The tree T_1 with root w_1^1 is now obtained by adding an edge for each i connecting w_i^1 and the root of X_i^1. Note that $T_1 \in \mathcal{G}_1$.

Let

$$\mathrm{Forb}(T_1, \dots, T_n) := \{G \in \mathcal{G}_1 \mid G \not\succcurlyeq T_1, \dots, T_n\}.$$

Similarly to the construction of T_1 define trees T_n, $n > 1$, recursively as follows.

Assuming that $\mathcal{G}_n := \mathrm{Forb}(T_1, \dots, T_{n-1})$ contains infinitely many topological types of trees, let $X_1^n, X_2^n, \dots \in \mathcal{G}_n$ be a choice of exactly two trees of each type represented in $\mathcal{G}_n$. Consider the disjoint countable union $X_1^n \cup X_2^n \cup \dots$ and let $R^n = w_1^n w_2^n \dots$ be an additional ray. Obtain T_n with root w_1^n by adding an edge for each i connecting w_i^n and the root of X_i^n.

Observe that if T_k exists and $T_k \in \mathcal{G}_k$, our construction yields the following:

- (i) T_k is universal among all trees in $\mathcal{G}_k$, since for any $X \in \mathcal{G}_k$ there is some j such that $X \preccurlyeq X_j^k \preccurlyeq T_k$;
- (ii) T_k contains two disjoint representatives X_m^k, X_n^k of its own topological type. Denote these as T_k^1 and T_k^2, let t_k^1 and t_k^2 be their roots, and let v_k be the first vertex on R^k adjacent to t_k^1 or t_k^2. By construction, $v_k < t_k^1, t_k^2$ in the tree-order of T_k, and v_k separates T_k^1 from T_k^2 in T_k; in particular, t_k^1 and t_k^2 are incomparable under the tree-order.

Lemma 2 *For all natural n, $\mathcal{G}_n$ contains infinitely many topological types of trees (so T_n exists), and $T_n \in \mathcal{G}_n$.*

PROOF. The assertion holds for $n = 1$. Let $n > 1$ and assume that T_k exists and $T_k \in \mathcal{G}_k$ for every $k < n$. Then (i) and (ii) apply to these trees T_k.

By definition of $\mathcal{G}_n$, every tree $T \in \mathcal{G}_1 \setminus \mathcal{G}_n$ satisfies $T \succcurlyeq T_k$ for some $k < n$. Choose k minimum. Then $T \in \mathrm{Forb}(T_1, \dots, T_{k-1}) = \mathcal{G}_k$, and hence $T \preccurlyeq T_k$ by (i), so T and T_k have the same topological type. Thus, every $T \in \mathcal{G}_1 \setminus \mathcal{G}_n$ belongs to one of finitely many topological types, those of $T_1, \dots, T_{n-1}$. Hence as $\mathcal{G}_1$ contains trees of infinitely many types, so does $\mathcal{G}_n$.

It remains to show that $T_n \in \mathcal{G}_n$. If not, then $T_n \succcurlyeq T_k$ for some $k < n$. By the induction hypothesis, T_k satisfies (ii). Given any (tree-order preserving) embedding of T_k into T_n, consider the images of t_k^1 and t_k^2 in T_n. Since incomparable vertices (with respect to the tree-order) map to incomparable vertices under such an embedding, (ii) implies that not both t_k^1 and t_k^2 map to vertices on R^n. Therefore one of them, t_k^i, maps into some X_j^n. Then T_k^i too maps into X_j^n, contradicting the fact that T_k^i has the same topological type as T_k but $X_j^n \in \mathrm{Forb}(T_1, \dots, T_k, \dots, T_{n-1})$. $\square$

From $T_n \in \mathrm{Forb}(T_1, \dots, T_{n-1})$, $n > 1$, we can now deduce that $T_1, T_2, \dots$ is a bad sequence. (It is in fact a descending sequence, since each T_n is universal in $\mathcal{G}_n$.) This contradicts Nash-Williams's theorem that the infinite trees are well-quasi-ordered under rooted topological embedding, and thus completes the proof of Theorem 1.

Concluding remarks Note that all the arguments used remain valid when restricting the class of trees to those trees every vertex of which has at most two successors. Thus, there are uncountably many topological types of these trees already.

A problem that remains open is to find a constructive proof of Theorem 1.

References

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