A CONSEQUENCE OF THE FACTORISATION THEOREM FOR POLYNOMIAL ORBITS ON NILMANIFOLDS – CORRIGENDUM TO *ACTA ARITH.* 154 (2012), 235–306

LILIAN MATTHIESEN

1. Introduction

The aim of this corrigendum is to prove a stronger version of [4, Theorem 16.4] since its original form does not fulfill all conditions required for an application in [4, §17]. The result in question is a consequence of the quantitative factorisation theorem for polynomial orbits on nilmanifolds that Green and Tao established in [3]. It has been devised as a replacement for the original (but in the setting of [4] non-applicable) factorisation theorem within the context of specific arithmetic applications of Green and Tao's *nilpotent Hardy–Littlewood method*, i.e. the machinery behind their celebrated work [2]. Apart from its application in [4], this result proved essential to the work in [1] and [5].

We briefly describe the differences between our consequence, Theorem 2.3 below, and Green and Tao's result. The factorisation theorem [3, Theorem 1.19], which we recall in §2, is a structure theorem which states (in quantitative terms) that an arbitrary polynomial sequence g on a nilmanifold may be decomposed as a product $\varepsilon g'\gamma$ of three polynomial sequences, where ε is slowly varying, g' is highly equidistributed on a certain submanifold, and γ is periodic. Unfortunately, the mutual dependencies of the parameters that quantify the respective properties of ε, g' and γ are too restrictive as to allow for a direct application of this result in the settings of [4, 1, 5]. Theorem 2.3 resolves this problem by essentially replacing upper bounds on integer parameters (e.g. the period of γ) by working with smooth numbers. This allows one to weaken the interdependence of the parameters controlling the level of equidistribution of g' and the bound on the period of γ. Combining property (3) from Theorem 2.3 with [4, Proposition 15.4], it is then possible to restrict the original polynomial sequence g to any subprogression of a rather large, but smooth, common difference without loosing the equidistribution property of g'.

To conclude this short introduction let us mention what goes wrong with the original application of [4, Theorem 16.4]. As in other applications of the factorisation theorem, one aims to exploit the fact that the sequences $n \mapsto g'_P(n)$, which replace the g' from before, are equidistributed. In order to access this property within the product sequence $n \mapsto \varepsilon_P(n)g'_P(n)\gamma_P(n)$, it is necessary to split the range of the variable n into subprogressions on which ε_P is almost constant and on which γ_P is constant. For this approach to work, it is crucial that g'_P is still equidistributed when restricted to these new subprogressions. Precisely this condition is, however, not guaranteed by [4, Theorem 16.4], as stated.

2. A new factorisation lemma for polynomial nilsequences

To start with, we recall the statement of the factorisation theorem from Green and Tao [3, Theorem 1.19], referring to their paper for any undefined terms and notation:

Theorem 2.1. *Let $m, d > 0$, and let $M_0, N > 1$ and $E > 0$ be real numbers. Suppose that G/Γ is an m-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d. Suppose that X is an M_0-rational Mal'cev basis $\mathscr{X}$ adapted to $G_\bullet$ and that $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$. Then there is an integer M with $M_0 \leqslant M \ll M_0^{O_{E,m,d}(1)}$, a rational subgroup $G' \subseteq G$, a Mal'cev basis $\mathscr{X}'$ for G'/Γ' in which each element is an M-rational combination of the elements of $\mathscr{X}$, and a decomposition $g = \varepsilon g' \gamma$ into polynomial sequences $\varepsilon, g', \gamma \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ with the following properties:*

(1) *$\varepsilon : \mathbb{Z} \to G$ is (M, N)-smooth;*
(2) *$g' : \mathbb{Z} \to G'$ takes values in G', and the finite sequence $(g(n)\Gamma')_{n \leqslant N}$ is totally $1/M^E$-equidistributed in G'/Γ', using the metric $d_{\mathscr{X}'}$ on G'/Γ';*
(3) *$\gamma : \mathbb{Z} \to G$ is M-rational, and $(\gamma(n)\Gamma)_{n \in \mathbb{Z}}$ is periodic with period at most M.*

We shall employ this result in an iterative process. To guarantee that this process terminates, we will ensure by appealing to the lemma below that in each application of the above result the rational subgroup G' is of strictly lower dimension than that of the ambient group G.

Lemma 2.2. *Under the hypotheses of Theorem 2.1, let $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$, and suppose that $g(n) = \varepsilon(n) g'(n) \gamma(n)$ is a factorisation which satisfies the conditions (1)–(3). Then there is a constant $C \geqslant 1$, only depending on m and d, such that whenever E is sufficiently large and $G' = G$, then g is totally $M^{-E/2C}$-equidistributed.*

Proof. Let $C \geqslant 1$ be a constant that will be determined in the course of the proof, and let $P \subseteq \{1, \ldots, N\}$ be a progression of length at least $M^{-E/2C} N$. Since the period of γ is bounded above by M, we may split P into at most M subprogressions, each of length at least $M^{-(E/(2C)+1)} N$, on which γ is constant. Next, we split each of these subprogressions into pieces of diameter between $M^{-((E/(2C)+1)} N$ and $2M^{-(E/(2C)+1)} N$. Let $\mathscr{P}$ denote the collection of all the resulting bounded diameter pieces. If $F : G/\Gamma \to \mathbb{C}$ is a Lipschitz function, then the right-invariance of the metric $d_{\mathscr{X}}$ (cf. [3, Definition 2.2]) implies for any n, n' that belong to the same element Q of $\mathscr{P}$ that:

$$\begin{aligned} |F(\varepsilon(n)g'(n)\gamma(n)\Gamma) - F(\varepsilon(n')g'(n)\gamma(n)\Gamma)| &\leqslant \|F\|_{\mathrm{Lip}}\ d_{\mathscr{X}}(\varepsilon(n)g'(n)\gamma(n), \varepsilon(n')g'(n)\gamma(n)) \\ &= \|F\|_{\mathrm{Lip}}\ d_{\mathscr{X}}(\varepsilon(n), \varepsilon(n')) \\ &\leqslant \|F\|_{\mathrm{Lip}}\ |n - n'| M/N \\ &\leqslant 2\|F\|_{\mathrm{Lip}}\ M^{-E/(2C)}. \end{aligned}$$

This estimate allows one to fix for any $Q \in \mathscr{P}$ the contribution of ε. To see this, choose for every $Q \in \mathscr{P}$ a fixed element and denote it s_Q. Then

$$\sum_{n \in Q} F(g(n)\Gamma) = \sum_{n \in Q} F(\varepsilon(s_Q) g'(n)\gamma(n)\Gamma) + O(\#Q \|F\|_{\mathrm{Lip}} M^{-E/(2C)}).$$

Let $H_Q : G/\Gamma \to \mathbb{C}$ denote the map $H_Q(h) := F(\varepsilon(s_Q)h\Gamma)$, and observe that the approximate left-invariance of $d_{\mathscr{X}}$ (cf. [3, Lemma A.5]) implies that $\|H_Q\|_{\mathrm{Lip}} \leqslant M_0^{O(1)}\|F\|_{\mathrm{Lip}}$. We furthermore have $\int_{G/\Gamma} F = \int_{G/\Gamma} H_Q$. The fact that $(g'(n)\Gamma)_{n\leqslant N}$ is totally M^{-E}-equidistributed in G/Γ implies a similar property for each of the sequences $(g'(n)\gamma(m)\Gamma)_{n\leqslant N}$ for fixed m. Indeed, it follows from [4, Proposition 14.3], which is a consequence of [3, Theorem 2.9], that there is a constant $C' = BB' > 1$, only depending on m and d, such that $(g'(n)\gamma(m)\Gamma)_{n\leqslant N}$ is totally $M^{-E/C'}$-equidistributed. We set $C = C'$. Making use of this equidistribution property, we obtain

$$\sum_{n\in Q} F(\varepsilon(s_Q)g'(n)\gamma(n)\Gamma) = \sum_{n\in Q} F(\varepsilon(s_Q)g'(n)\gamma(n_Q)\Gamma)$$
$$= \left(\int_{G/\Gamma} F + O\left(M_0^{O(1)}M^{-E/C}\|F\|_{\mathrm{Lip}}\right)\right)\#Q$$

for any $Q \in \mathscr{P}$, and, hence,

$$\sum_{n\in N} F(g(n)\Gamma) = N\left(\int_{G/\Gamma} F + \|F\|_{\mathrm{Lip}} O\left(M^{-E/(2C)} + M_0^{O(1)}M^{-E/C}\right)\right).$$

This completes the proof. □

The following generalisation of [4, Theorem 16.4] is the main result of this paper.

Theorem 2.3 (Factorisation lemma)**.** *Let N and $T = T(N)$ be positive integer parameters that satisfy $N^{1-\varepsilon} \ll_\varepsilon T \ll N$ and let $k : \mathbb{N} \to \mathbb{N}$ be a slowly growing function. Let m, d, B, E and $Q_0 \geqslant 1$ be positive integers. Suppose that G/Γ is an m-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d. Suppose that $\mathscr{X}$ is a Q_0-rational Mal'cev basis adapted to $G_\bullet$, and that $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$. Suppose further that $Q_0 \leqslant \log k(N)$. Let $R = R(N)$ be a parameter that satisfies $R \geqslant Q_0$ and $R(N)^t \ll_t N$ for all $t > 0$. Then there is an integer Q with $Q_0 \leqslant Q \ll Q_0^{O_{B,m,d}(1)}$ and a partition of $\{1, \dots, T\}$ into at most $R^{O_{m,d,B,E}(1)}$ disjoint subprogressions P, each of $k(N)$-smooth common difference $q(P) \ll R^{O_{m,d,B,E}(1)}$ and each of length $T/q(P) + O(1)$, such that the restriction $(g(n))_{n\in P}$ of g to any of the progressions P can be factorised as follows.*

There is a rational subgroup $G' \leqslant G$, depending on P, and a Mal'cev basis $\mathscr{X}'$ for $G'\Gamma/\Gamma$ such that every element of $\mathscr{X}'$ is a Q-rational combination of elements from $\mathscr{X}$ (that is, each coefficient is rational of height bounded by Q). Suppose $P = \{qn + r : 1 \leqslant n \leqslant T/q + O(1)\}$, where $q = q(P)$, then we have a factorisation

$$g(qn + r) = \varepsilon_P(n)g'_P(n)\gamma_P(n)\ ,$$

where $\varepsilon_P, g'_P, \gamma_P$ are polynomial sequences from $\mathrm{poly}(\mathbb{Z}, G_\bullet)$ *with the properties*

(1) *$\varepsilon_P : \mathbb{Z} \to G$ is $(Q, T/q)$-smooth;*
(2) *$\gamma_P : \mathbb{Z} \to G$ arises as the product of at most m Q-rational polynomial sequences and the sequence $(\gamma_P(n)\Gamma)_{n\in\mathbb{Z}}$ is periodic with a $k(N)$-smooth period $q_{\gamma_P} \leqslant Q$;*
(3) *$g'_P : \mathbb{Z} \to G'$ takes values in G' and for each $k(N)$-smooth number $\tilde{q} < (qq_{\gamma_P}R)^E$ the finite sequence $(g'_P(\tilde{q}n)\Gamma')_{n\leqslant T/(q\tilde{q})}$ is totally Q^{-B}-equidistributed in $G'\Gamma/\Gamma$.*

Our proof requires the fact that a polynomial sequence that fails to be totally equidistributed also fails to be equidistributed when allowing polynomial changes in the equidistribution parameter. This is made precise in [6, Lemma 6.2], which we restate here for simplicity:

Lemma 2.4. *Let N be a positive integer and let $\delta : \mathbb{N} \to [0,1]$ be a function that satisfies $\delta(x)^{-t} \ll_t x$ for all $t > 0$. Suppose that G/Γ is an m-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d, and suppose that G has a $\frac{1}{\delta(N)}$-rational Mal'cev basis adapted to $G_\bullet$. Then there is $1 \leqslant C \ll_{d,m} 1$ such that the following holds. Let $E > C$ be real, and suppose that $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ is a polynomial sequence such that $(g(n)\Gamma)_{n \leqslant N}$ is $\delta(N)^E$-equidistributed. Then $(g(n)\Gamma)_{n\leqslant N}$ is totally $\delta(N)^{E/C}$-equidistributed, provided N is sufficiently large.*

Proof of Lemma 2.3 assuming Lemma 2.4. We may suppose that g does not satisfy (3) with Q replaced by Q_0. That is, there is some $k(N)$-smooth integer $q_1 \leqslant R^E$ such that $(g(q_1 n)\Gamma)_{n\leqslant T/q_1}$ fails to be totally Q_0^{-B}-equidistributed. By Lemma 2.4, this sequence also fails to be Q_0^{-BC}-equidistributed for some $C = O_{m,d}(1)$. Writing $z_1 := (q_1)^d$, we deduce from [4, Lemma 16.3] that each of the sequences $(g(z_1 n + r_1)\Gamma)_{n\leqslant T/z_1}$ with $0 \leqslant r_1 < z_1$ fails to be $Q_0^{-BCC'}$-equidistributed in G/Γ for some $C' = O_{m,d}(1)$. Now, we run through all $0 \leqslant r_1 < z_1$ in turn.

Applying Theorem 2.1 and Lemma 2.2 to any of these sequences yields some $Q_0 \leqslant Q_1 \ll Q_0^{O(B,m,d)}$, a Q_1-rational subgroup $G_1 < G$ of dimension strictly smaller than that of G, and a factorisation

$$g(z_1 n + r_1) = \varepsilon_{r_1}(n) g'_{r_1}(n) \gamma_{r_1}(n),$$

where the finite sequence $(g'_{r_1}(n)\Gamma_1)_{n\leqslant T/z_1}$ is totally Q_1^{-B}-equidistributed in

$$G_1/\Gamma_1 := G_1/(\Gamma \cap G_1),$$

where $(\varepsilon_{r_1}(n)\Gamma)_{n\in\mathbb{Z}}$ is $(Q_1, T/z_1)$-smooth, and where $(\gamma_{r_1}(n)\Gamma)_{n\in\mathbb{Z}}$ is periodic with period at most Q_1.

If g'_{r_1} is totally Q_1^{-B}-equidistributed on every subprogression $\{n \equiv 0 \pmod{q_2}\}$ of $k(N)$-smooth common difference $q_2 < (z_1 Q_1 R)^E$, then we stop (and turn to the next choice of r_1). Otherwise, invoking Lemma 2.4 and [4, Lemma 16.3] again, there are positive integers $C, C' = O_{d,m}(1)$ and a $k(N)$-smooth integer q_2 as above such that, with $z_2 := q_2^d$, the finite sequence $(g_{r_1,r_2}(n))_{n\leqslant T/(z_1 z_2)}$ defined by $g_{r_1,r_2}(n) := g'_{r_1}(z_2 n + r_2)$ fails to be $Q_1^{-BCC'}$-equidistributed for every $0 \leqslant r_2 < z_2$. We proceed as before.

This process yields a tree of operations which has height at most $m = \dim G$, since each time the factorisation theorem is applied, a new sequence $g'_{r_1,\dots,r_i}$ is found that takes values in some strictly lower dimensional submanifold $G_i = G_i(r_1,\dots,r_i)$ of $G_{i-1}(r_1,\dots,r_{i-1})$. Thus, we can apply the factorisation theorem at most m times in a row before the manifold involved has dimension 0.

The tree we run through starts with g, which has z_1 neighbours g_{r_1}, one for each $0 \leqslant r_1 < z_1$. For each r_1, the vertex g_{r_1} has $z_2 = z_2(r_1)$ neighbours g_{r_1,r_2}, one for each $0 \leqslant r_2 < z_2(r_1)$, etc. As a result, we obtain a decomposition of the range $\{1,\dots,T\}$ into at most $R^{O_{m,d,B,E}(1)}$ subprogressions of the form

$$\begin{aligned} P &= \{z_1(z_2(z_3(\dots(z_t n + r_t)\dots) + r_3) + r_2) + r_1 \ : \ n \leqslant T/(z_1 z_2 \dots z_t)\} \\ &= \{z_1 z_2 \dots z_t n + r \ : \ n \leqslant T/(z_1 z_2 \dots z_t)\} \ , \end{aligned}$$

for $t \leqslant m$, some r, and where each z_i depends on $r_1,\dots,r_{i-1}$. The common difference of such a progression P is $k(N)$-smooth and bounded by $R^{O_{m,d,B,E}(1)}$. The iteration process furthermore yields a factorisation of $g_{r_1,\dots,r_t}$, which is the restriction of g to P:

$$g_{r_1,\dots,r_t}(m) = g(z_1 z_2 \dots z_t m + r) = \tilde{\varepsilon}_{r_1,\dots,r_t}(m) g'_t(m) \tilde{\gamma}_{r_1,\dots,r_t}(m) \ ,$$

where

$$\tilde{\varepsilon}_{r_1,\dots,r_t}(m) = \varepsilon_{r_1}(z_2 \dots z_t m + \tilde{r}_2) \dots \varepsilon_{r_1,\dots,r_{t-1}}(z_t m + \tilde{r}_t) \varepsilon_{r_1,\dots,r_t}(m)$$

for certain integers $\tilde{r}_2, \tilde{r}_3, \dots, \tilde{r}_t$, and

$$\tilde{\gamma}_{r_1,\dots,r_t}(m) = \gamma_{r_1,\dots,r_t}(m) \gamma_{r_1,\dots,r_{t-1}}(z_t m + \tilde{r}_t) \dots \gamma_{r_1}(z_2 \dots z_t m + \tilde{r}_2) \ .$$

The factor $\tilde{\varepsilon}_{r_1,\dots,r_t}(m)$ is a $(Q_0^{O_{B,d,m}(1)}, T/(z_1 \dots z_t))$-smooth sequence. This follows from the triangle inequality, the right-invariance and the approximate left-invariance of $d_{\mathscr{X}}$; we refer to the discussion following Definition 16.1 in [4] for details and to [3, App. A] for the properties of $d_{\mathscr{X}}$.

Since each $\gamma_{r_1,\dots,r_i}$ with $i \leqslant t$ is periodic with period at most $Q_0^{O_{m,d,B}(1)}$ and since $t \leqslant m$, we deduce that $\tilde{\gamma}_{r_1,\dots,r_t}$ is periodic with period at most $Q_0^{O_{m,d,B}(1)}$. The bound $Q_0 \leqslant \log k(N)$ implies that this period is $k(N)$-smooth provided N is sufficiently large.

Finally, g'_t satisfies property (3) by construction. □

Remark concerning Theorems 15.2 and 15.4 from [4]**.** The conclusions of these two results, the latter of which was mentioned in the introduction, can be simplified. Indeed, the conclusions imply that $(g \circ P(n) \pmod{\mathbb{Z}})_{n \in [(N/\gamma_{d'})^{1/d'}]}$ (resp. $(g \circ P(n)\Gamma)_{n \in [(N/\gamma_{d'})^{1/d'}]}$) is totally $\delta^{1/O_{d,d'}(1)}$-equidistributed in $\mathbb{R}/\mathbb{Z}$ (resp. G/Γ), provided N is sufficiently large.

References

[1] T.D. Browning and L. Matthiesen, Norm forms for arbitrary number fields as products of linear polynomials. arXiv:1307.7641.
[2] B.J. Green and T.C. Tao, Linear equations in primes. *Annals of Math.* **171** (2010), 1753–1850.
[3] B.J. Green and T.C. Tao, The quantitative behaviour of polynomial orbits on nilmanifolds. *Annals of Math.* **175** (2012), 465–540.
[4] L. Matthiesen, Linear correlations amongst numbers represented by positive definite binary quadratic forms. *Acta Arith.* **154** (2012), 235–306.
[5] L. Matthiesen, On the square-free representation function of a norm form and nilsequences. *J. Inst. Math. Jussieu*, to appear.
[6] L. Matthiesen, Generalized Fourier coefficients of multiplicative functions. `arXiv:1405.1018`.

Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover
E-mail address: `matthiesen@math.uni-hannover.de`